

# BADLY APPROXIMABLE POINTS ON CURVES AND UNIPOTENT ORBITS IN HOMOGENEOUS SPACES

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**ABSTRACT.** In this paper, we study the weighted  $n$ -dimensional badly approximable points on curves. Given an analytic non-degenerate curve  $\varphi : I = [a, b] \rightarrow \mathbb{R}^n$ , we will show that any countable intersection of the sets of the weighted badly approximable points on  $\varphi(I)$  has full Hausdorff dimension. This strengthens a result of Beresnevich [Ber15] by removing the condition on the weights. Compared with the work of Beresnevich, in this paper, we study the problem through homogeneous dynamics. It turns out that in order to solve this problem, it is crucial to study the distribution of long pieces of unipotent orbits in homogeneous spaces. The proof relies on the linearization technique and  $\mathrm{SL}(2, \mathbb{R})$  representations.

## 1. INTRODUCTION

**1.1. Badly approximable vectors.** Given a positive integer  $n$ , the weighted version of Dirichlet's approximation theorem says the following:

**Theorem 1.1** (Dirichlet's Theorem, 1842). *A vector  $\mathbf{r} = (r_1, \dots, r_n)$  is called a  $n$ -dimensional weight if  $r_i \geq 0$  for  $i = 1, \dots, n$  and*

$$r_1 + \dots + r_n = 1.$$

*For any  $n$ -dimensional weight  $\mathbf{r} = (r_1, \dots, r_n)$ , the following statement holds. For any vector  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$  and any  $N > 1$ , there exists an integer vector  $\mathbf{p} = (p_1, \dots, p_n, q) \in \mathbb{Z}^{n+1}$  such that  $0 < |q| \leq N$  and*

$$|qx_i + p_i| \leq N^{-r_i}, \text{ for } i = 1, \dots, n.$$

This theorem is the starting point of study in simultaneous Diophantine approximation. Using this theorem, one can easily show the following:

**Corollary 1.2.** *For any vector  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ , there are infinitely many integer vectors  $\mathbf{p} = (p_1, \dots, p_n, q) \in \mathbb{Z}^{n+1}$  with  $q \neq 0$  satisfying the following:*

$$(1.1) \quad |q|^{r_i} |qx_i + p_i| \leq 1 \text{ for } i = 1, \dots, n.$$

For almost every vector  $\mathbf{x} \in \mathbb{R}^n$ , the above corollary remains true if we replace 1 with any smaller constant  $c > 0$  on the right hand side of (1.1). The exceptional vectors are called  $\mathbf{r}$ -weighted badly approximable vectors. We give the formal definition as follows:

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**Definition 1.3.** Given a  $n$ -dimensional weight  $\mathbf{r} = (r_1, \dots, r_n)$ , a vector  $\mathbf{x} \in \mathbb{R}^n$  is called  $\mathbf{r}$ -weighted badly approximable if there exists a constant  $c > 0$  such that for any  $\mathbf{p} = (p_1, \dots, p_n, q) \in \mathbb{Z}^{n+1}$  with  $q \neq 0$ ,

$$\max_{1 \leq i \leq n} |q|^{r_i} |qx_i + p_i| \geq c.$$

For a  $n$ -dimensional weight  $\mathbf{r}$ , let us denote the set of  $\mathbf{r}$ -weighted badly approximable vectors in  $\mathbb{R}^n$  by  $\mathbf{Bad}(\mathbf{r})$ . In particular,  $\mathbf{Bad}(1)$  denotes the set of badly approximable numbers.

The study of the size of  $\mathbf{Bad}(\mathbf{r})$  has a long history and is active in both number theory and homogeneous dynamical systems. It is well known that the Lebesgue measure of  $\mathbf{Bad}(\mathbf{r})$  is zero for any  $n$ -dimensional weight  $\mathbf{r}$ . However, people have shown that  $\mathbf{Bad}(\mathbf{r})$  has full Hausdorff dimension, cf. [Jar28], [Sch66], [PV02] and [KW10].

For the intersection of sets of different weighted badly approximable vectors, Wolfgang M. Schmidt makes the following famous conjecture in 1982:

**Conjecture 1.4** (Schmidt's Conjecture, see [Sch83]).

$$\mathbf{Bad}(1/3, 2/3) \cap \mathbf{Bad}(2/3, 1/3) \neq \emptyset.$$

In 2011, Badziahin, Pollington and Velani [BPV11] settle this conjecture by showing the following: for any countable collection of 2-dimensional weights  $\{(i_t, j_t) : t \in \mathbb{N}\}$ , if  $\liminf_{t \rightarrow \infty} \min\{i_t, j_t\} > 0$ , then

$$\dim_H \left( \bigcap_{t=1}^{\infty} \mathbf{Bad}(i_t, j_t) \right) = 2.$$

Here  $\dim_H(\cdot)$  denotes the Hausdorff dimension of a set. An [An16] later strengthens their result by removing the condition on the weights. In fact, An proves the following much stronger result: for any 2-dimensional weight  $(r_1, r_2)$ ,  $\mathbf{Bad}(r_1, r_2)$  is  $(24\sqrt{2})^{-1}$ -winning. Here a set is called  $\alpha$ -winning if it is a winning set for Schmidt's  $(\alpha, \beta)$ -game for any  $\beta \in (0, 1)$ . This statement implies that any countable intersection of sets of weighted badly approximable vectors is  $\alpha$ -winning. The reader is referred to [Sch66] for more details of Schmidt's game.

For  $n \geq 3$ , Beresnevich [Ber15] proves the following theorem:

**Theorem 1.5** (see [Ber15, Corollary 1]). *Let  $n \geq 2$  be an integer and  $\mathcal{U} \subset \mathbb{R}^n$  be an analytic and non-degenerate submanifold in  $\mathbb{R}^n$ . Here a submanifold is called non-degenerate if it is not contained in any hyperplane of  $\mathbb{R}^n$ . Let  $W$  be a finite or countable set of  $n$ -dimensional weights such that  $\inf_{\mathbf{r} \in W} \{\tau(\mathbf{r})\} > 0$  where  $\tau(r_1, \dots, r_n) := \min\{r_i : r_i > 0\}$  for an  $n$ -dimensional weight  $(r_1, \dots, r_n)$ . Then*

$$\dim_H \left( \bigcap_{\mathbf{r} \in W} \mathbf{Bad}(\mathbf{r}) \cap \mathcal{U} \right) = \dim \mathcal{U}.$$

**1.2. Notation.** In this paper, we will fix the following notation.

For a set  $\mathcal{S}$ , let  $|\mathcal{S}|$  denote the cardinality of  $\mathcal{S}$ . For a measurable subset  $E \subset \mathbb{R}$ , let  $m(E)$  denote its Lebesgue measure.

For a matrix  $M$ , let  $M^T$  denote its transpose. For integer  $k > 0$ , let  $I_k$  denote the  $k$ -dimensional identity matrix.

Let  $\|\cdot\|$  denote the supremum norm on  $\mathbb{R}^n$  and  $\mathbb{R}^{n+1}$ . Let  $\|\cdot\|_2$  denote the Euclidean norm on  $\mathbb{R}^n$  and  $\mathbb{R}^{n+1}$ . For  $\mathbf{x} \in \mathbb{R}^{n+1}$  (or  $\in \mathbb{R}^n$ ) and  $r > 0$ , let  $B(\mathbf{x}, r)$  denote the closed ball in  $\mathbb{R}^{n+1}$  (or  $\mathbb{R}^n$ ) centered at  $\mathbf{x}$  of radius  $r$ , with respect to  $\|\cdot\|$ . For every  $i = 1, \dots, n+1$ , there is a natural supremum norm on  $\bigwedge^i \mathbb{R}^{n+1}$ . Let us denote it by  $\|\cdot\|$ .

Throughout this paper, when we say that  $c$  is a constant, we always mean that  $c$  is a constant only depending on the dimension  $n$ . For quantities  $A$  and  $B$ , let us use  $A \ll B$  to mean that there is a constant  $C > 0$  such that  $A \leq CB$ . Let  $A \asymp B$  mean that  $A \ll B$  and  $B \ll A$ . For a quantity  $A$ , let  $O(A)$  denote a quantity which is  $\ll A$  or a vector whose norm is  $\ll A$ .

**1.3. Main results.** In this paper, we will strengthen Theorem 1.5 by removing the condition on weights:

**Theorem 1.6.** *Let  $n \geq 2$  be an integer and  $\mathcal{U} \subset \mathbb{R}^n$  be an analytic and non-degenerate submanifold in  $\mathbb{R}^n$ . Let  $W$  be a finite or countable set of  $n$ -dimensional weights. Then*

$$\dim_H \left( \bigcap_{\mathbf{r} \in W} \mathbf{Bad}(\mathbf{r}) \cap \mathcal{U} \right) = \dim \mathcal{U}.$$

By the reduction argument in [Ber15], to prove the above theorem, it suffices to prove the theorem for analytic curves:

**Theorem 1.7.** *Let  $\varphi : I = [a, b] \rightarrow \mathbb{R}^n$  be an analytic and non-degenerate curve in  $\mathbb{R}^n$ . Let  $W$  be a finite or countable set of  $n$ -dimensional weights. Then*

$$\dim_H \left( \bigcap_{\mathbf{r} \in W} \mathbf{Bad}(\mathbf{r}) \cap \varphi(I) \right) = 1.$$

By [Ber15], Theorem 1.6 has the following corollary:

**Corollary 1.8.** *Let  $m, n \in \mathbb{N}$ ,  $B$  be a ball in  $\mathbb{R}^m$ ,  $W$  be a finite or countable set of  $n$ -dimensional weights and  $\mathcal{F}_n(B)$  be a finite family of analytic non-degenerate maps  $\mathbf{f} : B \rightarrow \mathbb{R}^n$ . Then*

$$\dim_H \left( \bigcap_{\mathbf{f} \in \mathcal{F}_n(B)} \bigcap_{\mathbf{r} \in W} \mathbf{f}^{-1}(\mathbf{Bad}(\mathbf{r})) \right) = m.$$

Compared with [Ber15], in this paper, we study this problem through homogeneous dynamics and prove Theorem 1.7 using the linearization technique.

**1.4. Bounded orbits in homogeneous spaces.** Let us briefly recall the correspondence between Diophantine approximation and homogeneous dynamics. The reader may see [Dan84], [KM98] and [KW08] for more details.

Let  $G = \mathrm{SL}(n+1, \mathbb{R})$ , and  $\Gamma = \mathrm{SL}(n+1, \mathbb{Z})$ . The homogeneous space  $X = G/\Gamma$  can be identified with the space of unimodular lattices in  $\mathbb{R}^{n+1}$ . The point  $g\Gamma$  is identified with the lattice  $g\mathbb{Z}^{n+1}$ . For  $\epsilon > 0$ , let us define

$$K_\epsilon := \{\Lambda \in X : \Lambda \cap B(\mathbf{0}, \epsilon) = \{\mathbf{0}\}\}.$$

It is well known that every  $K_\epsilon$  is a compact subset of  $X$  and every compact subset of  $X$  is contained in some  $K_\epsilon$ .

For a weight  $\mathbf{r} = (r_1, \dots, r_n)$ , let us define the diagonal subgroup  $A_{\mathbf{r}} \subset G$  as follows:

$$A_{\mathbf{r}} := \left\{ a_{\mathbf{r}}(t) := \begin{bmatrix} e^{r_1 t} & & & \\ & \ddots & & \\ & & e^{r_n t} & \\ & & & e^{-t} \end{bmatrix} : t \in \mathbb{R} \right\}.$$

For  $\mathbf{x} \in \mathbb{R}^n$ , let us denote

$$V(\mathbf{x}) := \begin{bmatrix} I_n & \mathbf{x} \\ & 1 \end{bmatrix}.$$

**Proposition 1.9.**  $\mathbf{x} \in \mathbf{Bad}(\mathbf{r})$  if and only if  $\{a_{\mathbf{r}}(t)V(\mathbf{x})\mathbb{Z}^{n+1} : t > 0\}$  is bounded.

*Proof.* The proof is well known and standard. We give the proof here for completeness.

On the one hand, if  $\mathbf{x} \in \mathbf{Bad}(\mathbf{r})$ , then there exists a constant  $c > 0$  such that for any integer vector

$$\mathbf{p} = (p_1, \dots, p_n, q)^T$$

with  $q \neq 0$ , we have that  $\max_{1 \leq i \leq n} |q|^{r_i} |qx_i + p_i| \geq c$ . Now let us consider the lattice  $\Lambda(t) = a_{\mathbf{r}}(t)V(\mathbf{x})\mathbb{Z}^{n+1}$ . We claim that for any  $t > 0$ , every nonzero vector in  $\Lambda(t)$  has norm at least  $c$ . In fact, for any nonzero integer vector  $\mathbf{p} = (p_1, \dots, p_n, q)^T$ , we have that

$$a_{\mathbf{r}}(t)V(\mathbf{x})\mathbf{p} = (e^{r_1 t}(qx_1 + p_1), \dots, e^{r_n t}(qx_n + p_n), e^{-t}q)^T.$$

If  $q = 0$ , then the claim is obvious since

$$\|a_{\mathbf{r}}(t)V(\mathbf{x})\mathbf{p}\| = \max_{1 \leq i \leq n} e^{r_i t} |p_i| > 1.$$

Let us assume that  $q \neq 0$ . For  $e^t < |q|$ , we have that

$$\|a_{\mathbf{r}}(t)V(\mathbf{x})\mathbf{p}\| \geq e^{-t}|q| > 1.$$

For  $e^t \geq |q|$ , we have that

$$\begin{aligned} \|a_{\mathbf{r}}(t)V(\mathbf{x})\mathbf{p}\| &\geq \max_{1 \leq i \leq n} e^{r_i t} |qx_i + p_i| \\ &\geq \max_{1 \leq i \leq n} q^{r_i} |qx_i + p_i| \geq c. \end{aligned}$$

This proves one direction of the statement.

On the other hand, if  $\{a_{\mathbf{r}}(t)V(\mathbf{x})\mathbb{Z}^{n+1} : t > 0\}$  is bounded, we want to show that  $\mathbf{x} \in \mathbf{Bad}(\mathbf{r})$ . In fact, there exists a constant  $c > 0$  such that  $a_{\mathbf{r}}(t)V(\mathbf{x})\mathbb{Z}^{n+1} \in K_c$  for all  $t > 0$ . Then for any integer vector  $\mathbf{p} = (p_1, \dots, p_n, q)^T$  with  $q \neq 0$ , we have that

$$\|a_{\mathbf{r}}(t)V(\mathbf{x})\mathbf{p}\| \geq c, \text{ for any } t > 0.$$

Let  $t = t_0$  such that  $e^{t_0} = 2|q|/c$ . Then the above inequality tells that

$$\max_{1 \leq i \leq n} \xi^{r_i} |q|^{r_i} |qx_i + p_i| \geq c, \text{ where } \xi = 2/c.$$

Let  $\epsilon = \max_{1 \leq i \leq n} \xi^{r_i}$ , then the above inequality implies that

$$\max_{1 \leq i \leq n} |q|^{r_i} |qx_i + p_i| \geq c\epsilon^{-1}.$$

This proves the other direction. □

Therefore our main theorem is equivalent to saying that for any analytic submanifold  $\mathcal{U} \subset \mathbb{R}^n$  and any countable collection of one-parameter diagonal subgroups  $\{A_{\mathbf{r}_s} : s \in \mathbb{N}\}$ , the set of  $\mathbf{x} \in \mathcal{U}$  such that

$$\{a_{\mathbf{r}_s}(t)V(\mathbf{x})\mathbb{Z}^{n+1} : t > 0\}$$

is bounded for all  $s \in \mathbb{N}$  has full Hausdorff dimension.

The study of bounded trajectories under the action of diagonal subgroups in homogeneous spaces is a fundamental topic in homogeneous dynamics and has been active for decades. The basic set up of this type of problems is the following. Let  $G$  be a Lie group and  $\Gamma \subset G$  be a nonuniform lattice in  $G$ . Then  $X = G/\Gamma$  is a noncompact homogeneous space. Let  $A = \{a(t) : t \in \mathbb{R}\}$  be a one-dimensional diagonalizable subgroup and let  $\mathbf{Bd}(A)$  be the set of  $x \in X$  such that  $A^+x$  is bounded in  $X$ , where  $A^+ := \{a(t) : t > 0\}$ . Then one can ask whether  $\mathbf{Bd}(A)$  has full Hausdorff dimension. For a submanifold  $\mathcal{U} \subset X$ , one can also ask whether  $\mathbf{Bd}(A) \cap \mathcal{U}$  has Hausdorff dimension  $\dim \mathcal{U}$ .

In 1986, Dani [Dan86] studies the case where  $G$  is a semisimple Lie group with  $\mathbb{R}$ -rank one. In this case, he proves that for any non-quasi-unipotent one parameter subgroup  $A \subset G$ ,  $\mathbf{Bd}(A)$  has full Hausdorff dimension. His proof relies on Schmidt's game. In 1996, Kleinbock and Margulis [KM96] study the case where  $G$  is a semisimple Lie group and  $\Gamma$  is a irreducible lattice in  $G$ . In this case, they prove that  $\mathbf{Bd}(A)$  has full Hausdorff dimension for any non-quasi-unipotent subgroup  $A$ . Their proof is based on the mixing property of the action of  $A$  on  $X$ . Recently, An, Guan and Kleinbock study the case where  $G = \mathrm{SL}(3, \mathbb{R})$  and  $\Gamma = \mathrm{SL}(3, \mathbb{Z})$ . They prove that for any countable one-parameter diagonalizable one-parameter subgroups  $\{F_s : s \in \mathbb{N}\}$ , the intersection  $\bigcap_{s=1}^{\infty} \mathbf{Bd}(F_s)$  has full Hausdorff dimension. Their proof closely follows the argument in the work of An [An16] and uses a variation of Schmidt's game.

**1.5. The linearization technique.** In [Ber15], the proof relies on a very careful study of the distribution of integer points in  $\mathbb{R}^{n+1}$  and the argument is elementary.

In this paper, we study this problem through homogeneous dynamics and tackle the technical difficulties using the linearization technique. We study the Diophantine properties using homogeneous dynamics. It turns out that in order to get the Hausdorff dimension, it is crucial to study distributions of long pieces of unipotent orbits in the homogeneous space  $G/\Gamma$ . To be specific, for a particular long piece  $C$  of a unipotent orbit, we need to estimate the length of the part in  $C$  staying outside a large compact subset  $K$  of  $G/\Gamma$ . In homogeneous dynamics, the standard tool to study this type of problem is the linearization technique. The linearization technique is a standard and powerful technique in homogeneous dynamics. Using the linearization technique, we can transform a problem in dynamical systems to one in linear representations. Then we can study this problem using tools and results in representation theory.

Let us briefly describe the technical difficulty when we apply the linearization technique. Let  $\mathcal{V}$  be a finite dimensional linear representation of  $\mathrm{SL}(n+1, \mathbb{R})$  with a norm  $\|\cdot\|$  and  $\Gamma(\mathcal{V}) \subset \mathcal{V}$  be a fixed discrete subset of  $\mathcal{V}$ . Let  $U = \{u(r) : r \in \mathbb{R}\}$  be a one dimensional unipotent subgroup of  $G$ . Given a large number  $T > 1$ , our main task is to estimate the measure of  $r \in [-T, T]$  such that there exists  $v \in \Gamma(\mathcal{V})$  such that  $\|u(r)v\| \leq \epsilon$  where  $\epsilon > 0$  is a small number. By Dani non-divergence theorem (see [Dan84]), the measure is very small

compared with  $T$  given that for any such  $v \in \Gamma(\mathcal{V})$

$$\max\{\|u(r)v\| : r \in [-T, T]\} \geq \rho$$

where  $\rho > 0$  is some fixed number. The difficulty is that there exists some  $v \in \Gamma(\mathcal{V})$ , such that

$$\max\{\|u(r)v\| : r \in [-T, T]\} \leq \rho.$$

Let us call such intervals  $T$ -bad intervals. In this paper, we will use the representation theory to get some nice properties of such  $v$ 's. We then use these nice properties to show that in a longer interval, say  $[-T^2, T^2]$ , the number of  $T$ -bad intervals is  $\ll T^{1-\mu}$  for some constant  $\mu > 0$ . This result is sufficient to prove Theorem 1.7.

In this paper,  $\mathcal{V}$  is the canonical representation of  $\mathrm{SL}(n+1, \mathbb{R})$  on  $\bigwedge^i \mathbb{R}^{n+1}$  and  $\Gamma(\mathcal{V}) = \bigwedge^i \mathbb{Z}^{n+1} \setminus \{0\}$  where  $i = 1, \dots, n$ .

The main technical results in this paper are proved in §4, §5.3 and §5.4.

We refer the reader to [Rat91], [MT94], [MS95], [Sha09b], [Sha09a] and [LM14] for more applications of the linearization technique.

**1.6. The organization of the paper.** The paper is organized as follows:

- In §2, we will recall some basic facts on Diophantine approximation, linear representations and lattices in  $\mathbb{R}^{n+1}$ .
- In §3, we will recall a theorem on computing the Hausdorff dimension of Cantor like sets. We will also construct a Cantor-like covering of the set of weighted badly approximable points.
- In §4, we will prove two technical results on counting lattice points. This is one of main technical contributions in this paper. Our proof relies on the linearization technique and  $\mathrm{SL}(2, \mathbb{R})$  representations.
- In §5, we will finish the proof of Theorem 3.5. Our proof relies on the Kleinbock-Margulis non-divergence theorem (Theorem 5.1) and the linearization technique.

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## 2. PRELIMINARIES

**2.1. Dual form of approximation.** We first recall the following equivalent definition of  $\mathbf{Bad}(\mathbf{r})$ :

**Lemma 2.1** (see [Ber15, Lemma 1]). *Let  $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{R}^n$  be a weight and  $\mathbf{x} \in \mathbb{R}^n$ . The following statements are equivalent:*

- (1)  $\mathbf{x} \in \mathbf{Bad}(\mathbf{r})$ .
- (2) *There exists  $c > 0$  such that for any integer vector  $(p_1, \dots, p_n, q)$  such that  $q \neq 0$ , we have that*

$$\max_{1 \leq i \leq n} |q|^{r_i} |qx_i + p_i| \geq c.$$

(3) *There exists  $c > 0$  such that for any  $N \geq 1$ , the only integer solution  $(a_0, a_1, \dots, a_n)$  to the system*

$$|a_0 + a_1 x_1 + \dots + a_n x_n| < cN^{-1}, \quad |a_i| < N^{r_i} \text{ for all } 1 \leq i \leq n$$

*is  $a_0 = a_1 = \dots = a_n = 0$ .*

*Proof.* The reader is referred to [Mah39], [BPV11, Appendix] and [Ber15, Appendix A] for the proof.  $\square$

Later in this paper we will use the third statement as the definition of  $\mathbf{Bad}(\mathbf{r})$ .

Given a weight  $\mathbf{r} = (r_1, \dots, r_n)$ , let us define

$$D_{\mathbf{r}} := \left\{ d_{\mathbf{r}}(t) := \begin{bmatrix} e^t & & & \\ & e^{-r_1 t} & & \\ & & \ddots & \\ & & & e^{-r_n t} \end{bmatrix} : t \in \mathbb{R} \right\}.$$

For  $\mathbf{x} \in \mathbb{R}^n$ , let us define

$$U(\mathbf{x}) := \begin{bmatrix} 1 & \mathbf{x}^T \\ & \mathbf{I}_n \end{bmatrix}.$$

If we use the third statement in Lemma 2.1 as the definition of  $\mathbf{Bad}(\mathbf{r})$ , then it is easy to show that  $\mathbf{x} \in \mathbf{Bad}(\mathbf{r})$  if and only if  $U(\mathbf{x})\mathbb{Z}^{n+1} \in \mathbf{Bd}(D_{\mathbf{r}})$ . In fact, the statement can be proved using the same argument as in the proof of Proposition 1.9.

**2.2. The canonical representation.** Let  $V = \mathbb{R}^{n+1}$ . There is a canonical representation of  $G = \mathrm{SL}(n+1, \mathbb{R})$  on  $V$ . It induces a canonical representation of  $G$  on  $\bigwedge^i V$  for every  $i = 1, 2, \dots, n$ . For  $g \in G$  and

$$\mathbf{v} = \mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_i \in \bigwedge^i V,$$

$$g\mathbf{v} = (g\mathbf{v}_1) \wedge \dots \wedge (g\mathbf{v}_i).$$

For  $i = 1, \dots, n$ , let  $\mathbf{e}_i \in \mathbb{R}^n$  denote the vector with 1 in the  $i$ th component and 0 in other components.

Let us fix a basis for  $V$  as follows. Let  $\mathbf{w}_+ := (1, 0, \dots, 0)^T$ . For  $i = 1, \dots, n$ , let  $\mathbf{w}_i := (0, \dots, 1, \dots, 0)^T$  with 1 in the  $i+1$ st component and 0 in other components. Then  $\{\mathbf{w}_+, \mathbf{w}_1, \dots, \mathbf{w}_n\}$  is a basis for  $V$ . Let  $W$  denote the subspace of  $V$  spanned by  $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ . For  $j = 2, \dots, n$ , let  $W_j$  the subspace of  $W$  spanned by  $\{\mathbf{w}_j, \dots, \mathbf{w}_n\}$ .

Let us define

$$Z := \left\{ z = \begin{bmatrix} 1 & \\ & \mathfrak{k}(z) \end{bmatrix} : \mathfrak{k}(z) \in \mathrm{SO}(n) \right\}.$$

There is a canonical action of  $\mathrm{SO}(n)$  on  $\mathbb{R}^n$ . For  $\mathfrak{k} \in \mathrm{SO}(n)$  and  $\mathbf{x} \in \mathbb{R}^n$ , let us denote by  $\mathfrak{k} \cdot \mathbf{x}$  the canonical action of  $\mathfrak{k}$  on  $\mathbf{x}$ . It is straightforward to check that for  $z = \begin{bmatrix} 1 & \\ & \mathfrak{k}(z) \end{bmatrix} \in Z$  and  $\mathbf{x} \in \mathbb{R}^n$ ,

$$zU(\mathbf{x})z^{-1} = U(\mathfrak{k}(z) \cdot \mathbf{x}).$$

For any  $\mathbf{x} \in \mathbb{R}^n$ ,  $U(\mathbf{x})$  can be embedded into a subgroup  $\mathrm{SL}(2, \mathbf{x})$  of  $G$  isomorphic to  $\mathrm{SL}(2, \mathbb{R})$ . In this  $\mathrm{SL}(2, \mathbb{R})$  copy,  $U(\mathbf{x})$  corresponds to  $\begin{bmatrix} 1 & \|\mathbf{x}\|_2 \\ & 1 \end{bmatrix}$ . For  $r > 0$ , let  $\xi_{\mathbf{x}}(r) \in \mathrm{SL}(2, \mathbf{x})$  denote the element corresponding to  $\begin{bmatrix} r & \\ & r^{-1} \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R})$ .

Let us consider the representation of  $\mathrm{SL}(2, \mathbf{x})$  on  $V$ .

Let us first consider the case  $\mathbf{x} = \mathbf{e}_1$ . Let us denote

$$U_1 := \{u_1(r) := U(r\mathbf{e}_1) : r \in \mathbb{R}\},$$

and

$$\Xi_1 := \{\xi_1(r) := \mathrm{diag}\{r, r^{-1}, 1, \dots, 1\} : r > 0\}.$$

It is easy to see that  $\xi_1(r)\mathbf{w}_+ = r\mathbf{w}_+$ ,  $u_1(r)\mathbf{w}_+ = \mathbf{w}_+$ ,  $\xi_1(r)\mathbf{w}_1 = r^{-1}\mathbf{w}_1$ ,  $u_1(r)\mathbf{w}_1 = \mathbf{w}_1 + r\mathbf{w}_+$ , and for any  $\mathbf{w} \in W_2$ ,  $\mathbf{w}$  is fixed by  $\mathrm{SL}(2, \mathbf{e}_1)$ .

For general  $\mathbf{x} \in \mathbb{R}^n$ , we have that  $\mathbf{x} = \|\mathbf{x}\|_2 \mathbf{e}_1$  for some  $\mathbf{e}_1 \in \mathrm{SO}(n)$ . Then

$$\mathrm{SL}(2, \mathbf{x}) = z(\mathbf{e}_1)\mathrm{SL}(2, \mathbf{e}_1)z^{-1}(\mathbf{e}_1)$$

where  $z(\mathbf{e}_1) = \begin{bmatrix} 1 & \\ & \mathbf{e}_1 \end{bmatrix} \in Z$ . In particular, we have that

$$U(\mathbf{x}) = z(\mathbf{e}_1)u_1(\|\mathbf{x}\|_2)z^{-1}(\mathbf{e}_1)$$

and  $\xi_{\mathbf{x}}(r) = z(\mathbf{e}_1)\xi_1(r)z^{-1}(\mathbf{e}_1)$ . Since  $z(\mathbf{e}_1)\mathbf{w}_+ = \mathbf{w}_+$  and  $z(\mathbf{e}_1)W = W$ , we have that  $\xi_{\mathbf{x}}(r)\mathbf{w}_+ = r\mathbf{w}_+$ ,  $U(\mathbf{x})\mathbf{w}_+ = \mathbf{w}_+$ ,  $\xi_{\mathbf{x}}(r)z(\mathbf{e}_1)\mathbf{w}_1 = r^{-1}\mathbf{e}_1 \cdot \mathbf{w}_1$ ,  $U(\mathbf{x})z(\mathbf{e}_1)\mathbf{w}_1 = z(\mathbf{e}_1)\mathbf{w}_1 + \|\mathbf{x}\|_2\mathbf{w}_+$  and for any  $\mathbf{w} \in z(\mathbf{e}_1)W_2$ ,  $\mathbf{w}$  is fixed by  $\mathrm{SL}(2, \mathbf{x})$ .

Let us consider the action of  $\mathrm{SL}(2, \mathbf{x})$  on  $\bigwedge^i V$  for  $i = 2, \dots, n$ . Let us denote  $\mathbf{x} = \|\mathbf{x}\|_2 \mathbf{e}_1$  as above. For any  $\mathbf{w} \in \bigwedge^{i-1} z(\mathbf{e}_1)W_2$ , we have that

$$\xi_{\mathbf{x}}(r)((z(\mathbf{e}_1)\mathbf{w}_1) \wedge \mathbf{w}) = r^{-1}((z(\mathbf{e}_1)\mathbf{w}_1) \wedge \mathbf{w})$$

,

$$U(\mathbf{x})((z(\mathbf{e}_1)\mathbf{w}_1) \wedge \mathbf{w}) = (z(\mathbf{e}_1)\mathbf{w}_1) \wedge \mathbf{w} + \|\mathbf{x}\|_2(\mathbf{w}_+ \wedge \mathbf{w}),$$

$$\xi_{\mathbf{x}}(r)(\mathbf{w}_+ \wedge \mathbf{w}) = r(\mathbf{w}_+ \wedge \mathbf{w})$$

and

$$U(\mathbf{x})(\mathbf{w}_+ \wedge \mathbf{w}) = \mathbf{w}_+ \wedge \mathbf{w}.$$

For any  $\mathbf{w} \in \bigwedge^i z(\mathbf{e}_1)W_2$  and any  $\mathbf{w}' \in \bigwedge^{i-2} z(\mathbf{e}_1)W_2$ , we have that  $\mathbf{w}$  and  $\mathbf{w}_+ \wedge (z(\mathbf{e}_1)\mathbf{w}_1) \wedge \mathbf{w}'$  are fixed by  $\mathrm{SL}(2, \mathbf{x})$ .

**2.3. Lattices in  $\mathbb{R}^{n+1}$ .** In this subsection let us recall some basic facts on lattices and sublattices in  $\mathbb{R}^{n+1}$ .

For a discrete subgroup  $\Delta$  of  $\mathbb{R}^{n+1}$ , let  $\mathrm{Span}_{\mathbb{R}}(\Delta)$  denote the  $\mathbb{R}$ -span of  $\Delta$ .

Let  $\Lambda \in X = G/\Gamma$  be a unimodular lattice in  $\mathbb{R}^{n+1}$ . For  $i = 1, \dots, n+1$ , let  $\Lambda_i \subset \Lambda$  be a  $i$ -dimensional sublattice of  $\Lambda$ . We say that  $\Lambda_i$  is primitive if  $\mathrm{Span}_{\mathbb{R}}(\Lambda_i) \cap \Lambda = \Lambda_i$ .

Given a  $i$ -dimensional primitive sublattice  $\Lambda_i$  of  $\Lambda$ , let us choose a basis  $\{\mathbf{v}_1, \dots, \mathbf{v}_i\}$  of  $\Lambda_i$ . Let us denote  $d(\Lambda_i) = \|\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_i\|$ .

For  $j = 1, \dots, i$ , let

$$\lambda_j(\Lambda_i) := \inf\{r \geq 0 : B(\mathbf{0}, r) \text{ contains at least } j \text{ linearly independent vectors of } \Lambda_i\}.$$



By the Minkowski Theorem (see [Cas57]), we have the following:

$$(2.1) \quad \lambda_1(\Lambda_i) \cdots \lambda_i(\Lambda_i) \asymp d(\Lambda_i).$$

Moreover, there exists a basis (called Minkowski reduced basis) of  $\Lambda_i$ ,  $\{\mathbf{v}_j : j = 1, \dots, i\}$ , such that  $\|\mathbf{v}_j\| \asymp \lambda_j(\Lambda_i)$  for every  $j = 1, \dots, i$ .

We will need the following result on counting sublattices:

**Proposition 2.2.** *There exists a constant  $N > 1$  such that the following statement holds. For any  $0 < \epsilon < \rho$  and any  $i = 1, \dots, n$ , let  $\Lambda \in K_\epsilon$  be a unimodular lattice in  $\mathbb{R}^{n+1}$  such that  $\lambda_1(\Lambda) \geq \epsilon$ . For  $\rho > 0$ , let  $\mathcal{C}_i(\Lambda, \rho)$  denote the collection of  $i$ -dimensional primitive sublattices  $\Lambda_i$  of  $\Lambda$  with  $d(\Lambda_i) \leq \rho$ . Then we have that,*

$$|\mathcal{C}_i(\Lambda, 1)| \leq \epsilon^{-N}.$$

*Proof.* First note that there exists a constant  $N_1 > 1$  such that for any  $i = 1, \dots, n$  and  $\rho > 0$ ,

$$|\mathcal{C}_i(\mathbb{Z}^{n+1}, \rho)| \leq \rho^{N_1}.$$

It is a standard fact that there exists a constant  $N_2 > 1$  such that for any  $\Lambda \in K_\epsilon$ , there exists  $g \in \text{SL}(n+1, \mathbb{R})$  with  $\|g^{-1}\| \leq \epsilon^{-N_2}$  such that  $\Lambda = g\mathbb{Z}^{n+1}$ . Let us fix  $\rho > \epsilon$  and  $i = 1, \dots, n$ . Then for any  $\Lambda_i \in \mathcal{C}_i(\Lambda, 1)$ , then we have that  $g^{-1}\Lambda_i \subset \mathbb{Z}^{n+1}$  and

$$d(g^{-1}\Lambda_i) \leq \|g^{-1}\|^i d(\Lambda_i) \leq \epsilon^{-(n+1)N_2}.$$

Therefore, we have that

$$|\mathcal{C}_i(\Lambda, 1)| \leq |\mathcal{C}_i(\mathbb{Z}^{n+1}, \epsilon^{-(n+1)N_2})| \leq \epsilon^{-N}$$

where  $N = N_1 N_2 (n+1)$ .

This completes the proof.  $\square$

### 3. A CANTOR LIKE CONSTRUCTION

In this section, we will introduce a Cantor like construction which will help us to compute Hausdorff dimension.

**Definition 3.1** (See [Ber15, §5]). For an integer  $R > 0$  and a closed interval  $J \subset \mathbb{R}$ , let us denote by  $\mathbf{Par}_R(J)$  the collection of intervals obtained by dividing  $J$  into  $R$  equal closed subintervals. For a collection  $\mathcal{I}$  of closed intervals, let us denote

$$\mathbf{Par}_R(\mathcal{I}) := \bigcup_{I \in \mathcal{I}} \mathbf{Par}_R(I).$$

A sequence  $\{\mathcal{I}_q\}_{q \in \mathbb{N}}$  of collections of closed intervals is called a  $R$ -sequence if for every  $q \geq 1$ ,  $\mathcal{I}_q \subset \mathbf{Par}_R(\mathcal{I}_{q-1})$ . For a  $R$ -sequence  $\{\mathcal{I}_q\}_{q \in \mathbb{N}}$  and  $q \geq 1$ , let us define  $\hat{\mathcal{I}}_q := \mathbf{Par}_R(\mathcal{I}_{q-1}) \setminus \mathcal{I}_q$  and

$$\mathcal{K}(\{\mathcal{I}_q : q \in \mathbb{N}\}) := \bigcap_{q \in \mathbb{N}} \bigcup_{I_q \in \mathcal{I}_q} I_q.$$

Then every  $R$ -sequence  $\{\mathcal{I}_q\}_{q \in \mathbb{N}}$  gives a Cantor like subset  $\mathcal{K}(\{\mathcal{I}_q\}_{q \in \mathbb{N}})$  of  $\mathbb{R}$ .

For  $q \geq 1$  and a partition  $\{\hat{\mathcal{I}}_{q,p}\}_{0 \leq p \leq q-1}$  of  $\hat{\mathcal{I}}_q$ , let us define

$$d_q(\{\hat{\mathcal{I}}_{q,p}\}_{0 \leq p \leq q-1}) := \sum_{p=0}^{q-1} \left(\frac{4}{R}\right)^{q-p} \max_{I_p \in \mathcal{I}_p} F(\hat{\mathcal{I}}_{q,p}, I_p),$$

where  $F(\hat{\mathcal{I}}_{q,p}, I_p) := |\{I_q \in \hat{\mathcal{I}}_{q,p}, I_q \in I_p\}|$ . Let us define

$$d_q(\mathcal{I}_q) := \min_{\{\hat{\mathcal{I}}_{q,p}\}_{0 \leq p \leq q-1}} d_q(\{\hat{\mathcal{I}}_{q,p}\}_{0 \leq p \leq q-1}),$$

where  $\{\hat{\mathcal{I}}_{q,p}\}_{0 \leq p \leq q-1}$  runs over all possible partitions of  $\hat{\mathcal{I}}_q$ . Let us define

$$d(\{\mathcal{I}_q\}_{q \in \mathbb{N}}) := \max_{q \in \mathbb{N}} d_q(\mathcal{I}_q).$$

**Definition 3.2** (See [Ber15, §5]). For  $M > 1$  and a compact subset  $X \subset \mathbb{R}$ , we say that  $X$  is  $M$ -Cantor rich if for any  $\epsilon > 0$  and any integer  $R \geq M$ , there exists a  $R$ -sequence  $\{\mathcal{I}_q\}_{q \in \mathbb{N}}$  such that

$$\mathcal{K}(\{\mathcal{I}_q\}_{q \in \mathbb{N}}) \subset X$$

and  $d(\{\mathcal{I}_q\}_{q \in \mathbb{N}}) \leq \epsilon$ .

Our proof relies on the following two theorems:

**Theorem 3.3** (See [Ber15, Theorem 6]). *Any  $M$ -Cantor rich set  $X$  has full Hausdorff dimension.*

**Theorem 3.4** (See [Ber15, Theorem 7]). *Let  $I_0$  be a compact interval. Then any countable intersection of  $M$ -Cantor rich sets in  $I_0$  is  $M$ -Cantor rich.*

To show Theorem 1.6, it suffices to find a constant  $M > 1$  and show that for any weight  $\mathbf{r}$ ,  $\varphi^{-1}(\mathbf{Bad}(\mathbf{r}) \cap \varphi(I))$  is  $M$ -Cantor rich. We will determine  $M > 1$  later.

**Theorem 3.5.** *There exists a constant  $M > 1$  such that for any weight  $\mathbf{r}$ ,  $\varphi^{-1}(\mathbf{Bad}(\mathbf{r}) \cap \varphi(I))$  is  $M$ -Cantor rich.*

Our main task is to prove Theorem 3.5.

Let us fix  $R \geq M$ . We will show that for any  $\epsilon > 0$ , we can construct a  $R$ -sequence  $\{\mathcal{I}_q\}_{q \in \mathbb{N}}$  such that  $\mathcal{K}(\{\mathcal{I}_q\}_{q \in \mathbb{N}}) \subset \mathbf{Bad}(\mathbf{r})$  and  $d(\{\mathcal{I}_q\}_{q \in \mathbb{N}}) < \epsilon$ . We will follow the construction in [Ber15] with some modifications.

**Standing Assumption 3.6.** *Let us make some assumptions to simplify the proof.*

**A.1** *Without loss of generality, we may assume that  $r_1 \geq r_2 \geq \dots \geq r_n$ . We may also assume that  $r_n > 0$ . By [Ber15], if  $r_n = 0$ , we can reduce the problem to the  $n - 1$  dimensional case.*

**A.2** *Since*

$$\varphi = (\varphi_1, \dots, \varphi_n)^T : I \rightarrow \mathbb{R}^n$$

*is analytic and nondegenerate, we may assume that for any  $s \in I$  and any  $i = 1, \dots, n$ ,  $\varphi_i^{(1)}(s) \neq 0$ . If this is not the case, we can choose a closed subinterval  $I' \subset I$  satisfying this condition. Then since  $I$  is closed, there exist constants  $C_1 > c_1 > 0$  such that for any  $s \in I$  and any  $i = 1, \dots, n$ ,  $c_1 \leq |\varphi_i^{(1)}(s)| \leq C_1$ .*

Let us fix some notation. Let  $m > 0$  be a large integer which we will determine later. Let  $\kappa = R^{-m}$ . Let  $b > 0$  be such that  $b^{1+r_1} = R$ . For  $t > 0$ , let us denote

$$g_{\mathbf{r}}(t) := \begin{bmatrix} b^t & & & \\ & b^{-r_1 t} & & \\ & & \ddots & \\ & & & b^{-r_n t} \end{bmatrix}.$$

For  $i = 1, \dots, n$ , let  $\lambda_i = \frac{1+r_i}{1+r_1}$ . Then we have that  $1 = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . Without loss of generality, we may assume that  $m(I) = 1$ .

Let us define the  $R$ -sequence as follows. Let  $\mathcal{I}_0 = \{I\}$ . Suppose that we have defined  $\mathcal{I}_{q-1}$  for  $q \geq 1$  and every  $I_{q-1} \in \mathcal{I}_{q-1}$  is a closed interval of length  $R^{-q+1}$ . Let us define  $\mathcal{I}_q \subset \mathbf{Par}_R(\mathcal{I}_q)$  as follows. For any  $I_q \in \mathbf{Par}_R(\mathcal{I}_q)$ ,  $I_q \in \hat{\mathcal{I}}_q$  if and only if there exists  $s \in I_q$  such that  $g_{\mathbf{r}}(q)U(\varphi(s))\mathbb{Z}^{n+1} \notin K_\kappa$ . That is to say, there exists  $\mathbf{a} \in \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$  such that  $\|g_{\mathbf{r}}(q)U(\varphi(s))\mathbf{a}\| \leq \kappa$ . Let us define  $\mathcal{I}_q = \mathbf{Par}_R(\mathcal{I}_{q-1}) \setminus \hat{\mathcal{I}}_q$ . This finishes the construction of  $\{\mathcal{I}_q\}_{q \in \mathbb{N}}$ . It is easy to see that

$$\mathcal{K}(\{\mathcal{I}_q\}_{q \in \mathbb{N}}) \subset \mathbf{Bad}(\mathbf{r}).$$

We need to prove the following:

**Proposition 3.7.** *For any  $\epsilon > 0$ , there exists an integer  $m > 0$  such that the  $R$ -sequence  $\{\mathcal{I}_q\}_{q \in \mathbb{N}}$  constructed as above with  $\kappa = R^{-m}$  satisfies that*

$$(3.1) \quad d(\{\mathcal{I}_q\}_{q \in \mathbb{N}}) \leq \epsilon.$$

Let  $N > 1$  be the constant from Proposition 2.2. Let us give the partition  $\{\hat{\mathcal{I}}_{q,p}\}_{0 \leq p \leq q-1}$  of  $\hat{\mathcal{I}}_q$  for each  $q \in \mathbb{N}$ .

**Definition 3.8.** Let us fix a small constant  $0 < \rho < 1$ . We will modify the choice of  $\rho$  later in this paper according to the constants coming from our technical results. For  $q \leq 10^6 n^4 Nm$ , let us define  $\hat{\mathcal{I}}_{q,0} := \hat{\mathcal{I}}_q$  and  $\hat{\mathcal{I}}_{q,p} = \emptyset$  for other  $p$ 's.

For  $q > 10^6 n^4 Nm$  and  $l = 2000n^2 Nm$ , let  $p = q - 2l$ . Let us define  $\hat{\mathcal{I}}_{q,p}$  to be the collection of  $I_q \in \hat{\mathcal{I}}_q$  with the following property: there exists  $x \in I_q$  such that for any  $j = 1, \dots, n$  and any  $\mathbf{w} = \mathbf{w}_1 \wedge \dots \wedge \mathbf{w}_j \in \bigwedge^j \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$ ,

$$\max\{g_{\mathbf{r}}(q)U(\varphi(x'))\mathbf{w} : x' \in [x - R^{-q+l}, x + R^{-q+l}]\} \geq \rho^j.$$

Let  $\eta = \frac{1}{100n^2}$  and  $\eta' = \frac{\eta}{1+r_1}$ . For  $q > 10^6 n^4 Nm$  and  $2000n^2 Nm \leq l \leq 2\eta'q$ , let  $p = q - 2l$ . For  $i = 1, \dots, n$ , let us define  $\hat{\mathcal{I}}_{q,p}(i)$  to be the collection of  $I_q \in \hat{\mathcal{I}}_q$  satisfying that there exists  $s \in I_q$  and  $\mathbf{v} = \mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_i \in \bigwedge^i \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$  such that

$$\|g_{\mathbf{r}}(q)U(\varphi(s'))\mathbf{v}\| \leq \rho^i,$$

for any  $s' \in [s - R^{-q+l}, s + R^{-q+l}]$  and for any  $j = 1, \dots, n$  and any  $\mathbf{w} = \mathbf{w}_1 \wedge \dots \wedge \mathbf{w}_j \in \bigwedge^j \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$ ,

$$\max\{\|g_{\mathbf{r}}(q)U(\varphi(s'))\mathbf{w}\| : s' \in [s - R^{-q+l}, s + R^{-q+l}]\} \geq \rho^j.$$

Let us define  $\hat{\mathcal{I}}_{q,p} = \bigcup_{i=1}^n \hat{\mathcal{I}}_{q,p}(i)$ .

For  $i = 1, \dots, n$ , let us define  $\hat{\mathcal{I}}_{q,0}(i)$  to be the collection of  $I_q \in \hat{\mathcal{I}}_q$  satisfying that there exists  $s \in I_q$  and  $\mathbf{v} = \mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_i \in \bigwedge^i \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$  such that

$$\max\left\{\|g_{\mathbf{r}}(q)U(\varphi(s))\mathbf{v}\| : s' \in [s - R^{-q(1-2\eta')}, s + R^{-q(1-2\eta')}] \right\} \leq \rho^i.$$

Let us define  $\hat{\mathcal{I}}_{q,0} = \bigcup_{i=1}^n \hat{\mathcal{I}}_{q,0}(i)$ .

Let us define  $\hat{\mathcal{I}}_{q,p} := \emptyset$  for other  $p$ 's. It is easy to see that  $\{\hat{\mathcal{I}}_{q,p}\}_{0 \leq p \leq q-1}$  is a partition of  $\hat{\mathcal{I}}_q$ , cf. [Ber15, Proposition 3].

Besides the definition of  $\{\hat{\mathcal{I}}_{q,p}\}_{0 \leq p \leq q-1}$ , let us also introduce the notion of dangerous intervals and extremely dangerous intervals:

**Definition 3.9.** For  $q \geq 10^6 n^4 Nm$ ,  $1000n^2 Nm \leq l \leq \eta'q$ , and  $\mathbf{a} \in \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$ , the  $(q, l)$ -dangerous interval associated with  $\mathbf{a}$ , which is denoted by  $\Delta_{q,l}(\mathbf{a})$ , is a closed interval of the form  $\Delta_{q,l}(\mathbf{a}) = [x - R^{-q+l}, x + R^{-q+l}] \subset I$  such that  $I_q \subset \Delta_{q,l}(\mathbf{a})$  for some  $I_q \in \hat{\mathcal{I}}_q$  and

$$\max\{\|g_{\mathbf{r}}(q)U(\varphi(s'))\mathbf{a}\| : s' \in \Delta_{q,l}(\mathbf{a})\} = c\rho,$$

for some  $c \in [1/2, 1]$ .

For  $q \geq 10^6 n^4 Nm$  and  $\mathbf{a} \in \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$ , the  $q$ -extremely dangerous interval associated with  $\mathbf{a}$ , which is denoted by  $\Delta_q(\mathbf{a})$ , is a closed interval of the form  $\Delta_q(\mathbf{a}) = [x - R^{-q+l'}, x + R^{-q+l'}]$  with  $l' > \eta'q$  such that  $I_q \subset \Delta_q(\mathbf{a})$  for some  $I_q \in \hat{\mathcal{I}}_q$  and

$$\max\{\|g_{\mathbf{r}}(q)U(\varphi(s'))\mathbf{a}\| : s' \in [x - 2R^{-q+l'}, x + 2R^{-q+l'}]\} = c\rho$$

for some  $c \in [1/2, 1]$ .

*Remark 3.10.* Note that for any  $q \geq 10^6 n^4 Nm$ , there are only finitely many  $\mathbf{a}$ 's such that  $\Delta_{q,l}(\mathbf{a})$  or  $\Delta_q(\mathbf{a})$  exist.

#### 4. COUNTING DANGEROUS INTERVALS

In this section we will count dangerous intervals and extremely dangerous intervals.

**Proposition 4.1.** *Let  $q \geq 10^6 n^4 Nm$ ,  $1000n^2 Nm \leq l \leq \eta'q$  and  $p = q - 2l$ . For  $I_p \in \mathcal{I}_p$ , let  $\mathcal{D}_{q,l}(I_p)$  denote the collection of  $(q, l)$ -dangerous intervals which intersect  $I_p$ . Then for any  $I_p \in \mathcal{I}_p$ ,*

$$|\mathcal{D}_{q,l}(I_p)| \ll R^{(1-\frac{1}{10n})l}.$$

**Proposition 4.2.** *Let  $q \geq 10^6 n^4 Nm$ . Let  $E_q \subset I$  denote the union of  $q$ -extremely dangerous intervals contained in  $I$ . Then  $E_q$  can be covered by a collection of  $N_q$  closed intervals of length  $\delta_q$  and*

$$N_q \leq \frac{K_0(\rho^{n+1}b^{-\eta q})^\alpha}{\delta_q}$$

where  $\delta_q = R^{-q(1-\eta')}$ ,  $K_0 > 0$  is a constant, and  $\alpha = \frac{1}{(n+1)(2n-1)}$ .

Proposition 4.2 is a rephrase of the following theorem due to Bernik, Kleinbock and Margulis:

**Theorem 4.3** (See [Ber15, Proposition 2] and [BKM01, Theorem 1.4]). *Let  $q > 10^6 n^4 Nm$ . Let us define  $E_q \subset I$  to be the collection of  $s \in I$  satisfying that there exists  $\mathbf{a} = (a_0, a_1, \dots, a_n)^T \in \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$  such that  $|a_i| < \rho b^{r_i q}$  for  $i = 1, \dots, n$ ,  $|f(s)| < \rho b^{-q}$  and  $|f'(s)| < b^{(r_1-\eta)q}$  where*

$$f(x) = a_0 + a_1\varphi_1(x) + \dots + a_n\varphi_n(x).$$

*Then  $E_q$  can be covered by a collection  $\mathcal{E}_q$  of intervals such that*

$$m(\Delta) \leq \delta_q \text{ for all } \Delta \in \mathcal{E}_q,$$

*and*

$$|\mathcal{E}_q| \leq \frac{K_0(\rho^{n+1}b^{-\eta q})^\alpha}{\delta_q},$$

where  $K_0 > 0$  is a constant,  $\delta_q = R^{-q(1-\eta')}$  and  $\alpha = \frac{1}{(n+1)(2n-1)}$ .

The theorem we quote here is the version used in [Ber15] with some minor modifications. The original version proved in [BKM01] is more general.

*Proof of Proposition 4.2.* In fact, for every  $q$ -extremely dangerous interval  $\Delta_q(\mathbf{a})$  where  $l' \geq \eta'q$  and  $\mathbf{a} = (a_0, a_1, \dots, a_n)^T$ , we have that

$$(4.1) \quad \|g_{\mathbf{r}}(q)U(\varphi(x))\mathbf{a}\| \leq \rho$$

Let us fix  $x \in \Delta_q(\mathbf{a})$ . By direct computation, we have that

$$g_{\mathbf{r}}(q)U(\varphi(x))\mathbf{a} = (v_0(x), v_1(x), \dots, v_n(x))^T$$

where

$$v_0(x) = b^q(a_0 + a_1\varphi_1(x) + \dots + a_n\varphi_n(x)),$$

and  $v_i(x) = b^{-r_i q}a_i$  for  $i = 1, \dots, n$ . Following the notation in Theorem 4.3, let us denote

$$f(x) = a_0 + a_1\varphi_1(x) + \dots + a_n\varphi_n(x).$$

Then (4.1) implies that  $|a_i| \leq \rho b^{r_i q}$  for  $i = 1, \dots, n$ , and  $|f(x)| \leq \rho b^{-q}$ . Moreover, for any  $x' \in [x - R^{-q(1-\eta')}, x + R^{-q(1-\eta')}]$ , we have that

$$\|g_{\mathbf{r}}(q)U(\varphi(x'))\mathbf{a}\| \leq \rho.$$

By direct calculation, this implies that

$$|f(x')| \leq \rho b^{-q}$$

for any  $x' \in [x - R^{-q(1-\eta')}, x + R^{-q(1-\eta')}]$ . Let  $x' = x + rR^{-q(1-\eta')}$  for some  $r \in [-1, 1]$ . Then

$$f(x') = f(x) + f'(x)rR^{-q(1-\eta')} + O(R^{-2q(1-\eta')}).$$

Therefore, we have that for any  $r \in [-1, 1]$ ,

$$\begin{aligned} |f'(x)rR^{-q(1-\eta')}| &= |f(x') - f(x) - O(R^{-2q(1-\eta')})| \\ &\leq |f(x')| + |f(x)| + O(R^{-2q(1-\eta')}) \\ &\leq \rho b^{-q} + \rho b^{-q} + \rho b^{-q} \leq b^{-q}. \end{aligned}$$

By letting  $r = 1$ , we have that

$$|f'(x)| \leq R^{q(1-\eta')}b^{-q} = b^{q(r_1-\eta')}.$$

The last equality above holds because  $b^{1+r_1} = R$  and  $\eta' = \frac{\eta}{1+r_1}$ . This shows that  $x \in E_q$  for any  $x \in \Delta_q(\mathbf{a})$ , i.e.,  $\Delta_q(\mathbf{a}) \subset E_q$ . Therefore, we have that  $D_q \subset E_q$ . Then the conclusion follows from Theorem 4.3.  $\square$

The rest of the section is devoted to the proof of Proposition 4.1. This is one of the main technical results of this paper.

*Proof of Proposition 4.1.* Let us fix  $I_p \in \mathcal{I}_p$ . Let  $I_p = [x - R^{-q+2l}, x + R^{-q+2l}]$ . We claim that we may assume that  $\varphi(I_p)$  is a straight line. In fact, for any  $x' \in I_p$ , let us write  $x' = x + rR^{-q+2l}$  for some  $r \in [-1, 1]$ . By Taylor's expansion, we have that

$$\begin{aligned} g_{\mathbf{r}}(q)U(\varphi(x')) &= g_{\mathbf{r}}(q)U(\varphi(x) + rR^{-q+2l}\varphi^{(1)}(x) + O(R^{-2q+4l})) \\ &= g_{\mathbf{r}}(q)U(O(R^{-2q+4l}))g_{\mathbf{r}}(-q)g_{\mathbf{r}}(q)U(\varphi(x) + rR^{-q+2l}\varphi^{(1)}(x)) \\ &= U(O(R^{-q+4l}))g_{\mathbf{r}}(q)U(\varphi(x) + rR^{-q+2l}\varphi^{(1)}(x)). \end{aligned}$$

Since  $l \leq \eta'q$ , we have that  $O(R^{-q+4l})$  is exponentially small. Therefore, we may assume that  $\varphi(x') = \varphi(x) + (x' - x)R^{-q+2l}\varphi^{(1)}(s)$  for any  $x' \in I_p$ .

Let us take a typical  $(q, l)$ -dangerous interval  $\Delta_{q,l}(\mathbf{a})$  that intersects  $I_p$ . Let us take  $x \in \Delta_{q,l}(\mathbf{a}) \cap I_p$  such that  $x \in I_{q-1}$  for some  $I_{q-1} \in \mathcal{I}_{q-1}$ . It is easy to see that either  $[x, x + R^{-q+l}] \subset \Delta_{q,l}(\mathbf{a})$ , or  $[x - R^{-q+l}, x] \subset \Delta_{q,l}(\mathbf{a})$ . Without loss of generality, we may assume that  $[x, x + R^{-q+l}] \subset \Delta_{q,l}(\mathbf{a})$ . Let us write  $\mathbf{a} = (a_0, a_1, \dots, a_n)^T \in \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$ . For  $x' \in [x, x + R^{-q+l}]$ , let us denote

$$g_{\mathbf{r}}(q)U(\varphi(x'))\mathbf{a} = \mathbf{v}(x') = (v_0(x'), v_1(x'), \dots, v_n(x'))^T.$$

Then we have that  $\max\{\|\mathbf{v}(x')\| : x' \in [x, x + R^{-q+l}]\} = c\rho$  for some  $c \in [1/2, 1]$ .

Recall that for  $j = 1, \dots, n$ ,  $\lambda_j = \frac{1+r_j}{1+r_1}$ . Let  $1 \leq n' \leq n$  be the largest index  $j$  such that  $(1 - \lambda_j)q \leq l$ .

For  $x' \in [x, x + R^{-q+l}]$ , let us write  $x' = x + rR^{-q+l}$  for  $r \in [0, 1]$ . By our assumption we have that  $\varphi(x') = \varphi(x) + rR^{-q+l}\varphi^{(1)}(x)$ . Let us write

$$\varphi^{(1)}(x) = (\varphi_1^{(1)}(x), \dots, \varphi_n^{(1)}(x))^T.$$

By our standing assumption on  $\varphi$  (Standing Assumption **A.2**), we have that  $c_1 \leq |\varphi_j^{(1)}(x)| \leq C_1$  for  $j = 1, \dots, n$ . By direct calculation, we have that

$$\begin{aligned} g_{\mathbf{r}}(q)U(\varphi(x'))\mathbf{a} &= g_{\mathbf{r}}(q)U(\varphi(x') - \varphi(x))g_{\mathbf{r}}(-q)g_{\mathbf{r}}(q)U(\varphi(x))\mathbf{a} \\ &= g_{\mathbf{r}}(q)U(rR^{-q+l}\varphi^{(1)}(x))g_{\mathbf{r}}(-q)\mathbf{v}(x) \\ &= g_{\mathbf{r}}(q)U(rR^{-q+l}\varphi^{(1)}(x))g_{\mathbf{r}}(-q)\mathbf{v}(x). \end{aligned}$$

Let us write

$$g_{\mathbf{r}}(q)U(rR^{-q+l}\varphi^{(1)}(x))g_{\mathbf{r}}(-q)\mathbf{v}(x) = \mathbf{v}(x') = (v_0(x'), v_1(x'), \dots, v_n(x'))^T,$$

where  $x' = x + rR^{-q+l}$ . By direct calculation, we have that

$$g_{\mathbf{r}}(q)U(rR^{-q+l}\varphi^{(1)}(x))g_{\mathbf{r}}(-q) = U\left(rR^l \sum_{i=1}^n R^{-(1-\lambda_i)q} \varphi_i^{(1)}(x) \mathbf{e}_i\right).$$

By our assumption, for  $i \geq n' + 1$ , we have that  $|rR^l R^{-(1-\lambda_i)q}| \leq 1$ . Therefore, if we write

$$U\left(-rR^l \sum_{i=n'+1}^n R^{-(1-\lambda_i)q} \varphi_i^{(1)}(x) \mathbf{e}_i\right) \mathbf{v}(x') = \mathbf{v}'(x') = (v'_0(x'), v'_1(x'), \dots, v'_n(x'))^T,$$

where  $v'_0(x') = v_0(x') - r \sum_{i=n'+1}^n R^l R^{-(1-\lambda_i)q} \varphi_i^{(1)}(x) v_i(x')$  and  $v'_i(x') = v_i(x')$  for  $i = 1, \dots, n$ . Then  $|v'_0(x')| < C = (n+1)C_1\rho$ , and  $|v'_i(x')| < \rho$  for  $i = 1, \dots, n$ . Let

$$\mathbf{h} = \sum_{i=1}^{n'} R^{-(1-\lambda_i)q} \varphi_i^{(1)}(x) \mathbf{e}_i$$

and

$$\mathbf{h}_W = \sum_{i=1}^{n'} R^{-(1-\lambda_i)q} \varphi_i^{(1)}(x) \mathbf{w}_i \in W.$$

Then  $\|\mathbf{h}\|_2 = \|\mathbf{h}_W\|_2 \asymp 1$ . For  $r \in [-1, 1]$ , and  $x' = x + rR^{-q+l}$ , our discussion above shows that

$$U(rR^l \mathbf{h})\mathbf{v}(x) = (v'_0(x'), v'_1(x'), \dots, v'_n(x'))^T,$$

where  $|v'_0(x')| < C$ , and  $|v'_i(x')| < \rho$  for  $i = 1, \dots, n$ . Let  $E_{n'}$  be the subspace of  $\mathbb{R}^n$  spanned by  $\{\mathbf{e}_1, \dots, \mathbf{e}_{n'}\}$  and  $W'_{n'}$  be the subspace of  $W$  spanned by  $\{\mathbf{w}_1, \dots, \mathbf{w}_{n'}\}$ . Then  $\mathbf{h} \in E_{n'}$ . Let

$\mathfrak{k} \in \text{SO}(n)$  be the element such that  $\mathfrak{k} \cdot \mathbf{e}_1 = \mathbf{h}$ ,  $\mathfrak{k} \cdot E_{n'} = E_{n'}$ , and  $\mathfrak{k} \cdot \mathbf{e}_i = \mathbf{e}_i$  for  $i = n' + 1, \dots, n$ . Let  $z(\mathfrak{k}) = \begin{bmatrix} 1 & \\ & \mathfrak{k} \end{bmatrix} \in Z$ . It is easy to see that  $z(\mathfrak{k})\mathbf{w}_+ = \mathbf{w}_+$ ,  $z(\mathfrak{k})\mathbf{w}_1 = \mathbf{h}_W$ ,  $z(\mathfrak{k})W'_{n'} = W'_{n'}$ , and  $z(\mathfrak{k})\mathbf{w}_i = \mathbf{w}_i$  for  $i = n' + 1, \dots, n$ . By the definition of  $z(\mathfrak{k})$  and our discussion in §2.2, we have that  $U(\mathbf{h}) = z(\mathfrak{k})U(\|\mathbf{h}\|_2 \mathbf{e}_1)z^{-1}(\mathfrak{k})$ . Therefore, we have that  $U(\mathbf{h})\mathbf{h}_W = \mathbf{h}_W + \|\mathbf{h}\|_2 \mathbf{w}_+$ . Moreover, we have that  $U(\mathbf{h})\mathbf{w}_+ = \mathbf{w}_+$ ; for  $i = 2, \dots, n'$ ,  $U(\mathbf{h})z(\mathfrak{k})\mathbf{w}_i = z(\mathfrak{k})\mathbf{w}_i$ ; and for  $i = n' + 1, \dots, n$ ,  $U(\mathbf{h})\mathbf{w}_i = \mathbf{w}_i$ . Let us write

$$\mathbf{v}(x) = a_+(x)\mathbf{w}_+ + \sum_{i=1}^{n'} a_i(x)z(\mathfrak{k})\mathbf{w}_i + \sum_{i=n'+1}^n a_i(x)\mathbf{w}_i.$$

Then the above discussion shows that

$$U(rR^l \mathbf{h})\mathbf{v}(x) = (a_+(x) + rR^l a_1(x))\mathbf{w}_+ + \sum_{i=1}^{n'} a_i(x)z(\mathfrak{k})\mathbf{w}_i + \sum_{i=n'+1}^n a_i(x)\mathbf{w}_i.$$

By our previous argument, we have that there exists a constant  $C > 0$  such that  $|a_i(x)| < C$  for  $i = 1, \dots, n$  and  $|a_+(x) + rR^l a_1(x)| < C$  for any  $r \in [0, 1]$ . This implies that  $|a_+(x)| < C$ , and  $|a_1(x)| < CR^{-l}$ . Therefore, we have that  $\mathbf{v}(x) \in z(\mathfrak{k})([-C, C] \times [-CR^{-l}, CR^{-l}] \times [-C, C]^{n-1})$ .

Now let us estimate  $|\mathcal{D}_{q,l}(I_p)|$ .

Suppose that  $\mathcal{D}_{q,l}(I_p) = \{\Delta_{q,l}(\mathbf{a}_u) : 1 \leq u \leq L\}$ . For each  $u = 1, \dots, L$ , let us take  $x_u \in \Delta_{q,l}(\mathbf{a}_u) \cap I_p$  such that  $x_u \in I_{q-1,u}$  for some  $I_{q-1,u} \in \mathcal{I}_{q-1}$ . Let us denote

$$\mathbf{v}_u = g_{\mathbf{r}}(q)U(\varphi(x_u))\mathbf{a}_u.$$

Then by our previous argument, we have that

$$\mathbf{v}_u = a_{u,+}\mathbf{w}_+ + \sum_{i=1}^{n'} a_{u,i}z(\mathfrak{k})\mathbf{w}_i + \sum_{i=n'+1}^n a_{u,i}\mathbf{w}_i,$$

where  $|a_{u,+}| < C$ ,  $|a_{u,1}| < CR^{-l}$ , and  $|a_{u,i}| < C$  for  $i = 2, \dots, n$ .

Now let us consider  $g_{\mathbf{r}}(q)U(\varphi(x_1))\mathbf{a}_u$ . Let us write  $x_u = x_1 - rR^{-q+2l}$  for some  $r \in [-1, 1]$ . Using our assumption that  $\varphi(I_p)$  is a straight line, we have that

$$\begin{aligned} & g_{\mathbf{r}}(q)U(\varphi(x_1))\mathbf{a}_u \\ &= g_{\mathbf{r}}(q)U(\varphi(x_1) - \varphi(x_u))g_{\mathbf{r}}(-q)g_{\mathbf{r}}(q)U(\varphi(x_u))\mathbf{a}_u \\ &= g_{\mathbf{r}}(q)U(\varphi(x_1) - \varphi(x_u))g_{\mathbf{r}}(-q)\mathbf{v}_u \\ &= g_{\mathbf{r}}(q)U(rR^{-q+2l}\varphi^{(1)}(x))g_{\mathbf{r}}(-q)\mathbf{v}_u \\ &= U\left(rR^{2l}\sum_{i=1}^n R^{-(1-\lambda_i)q}\varphi_i^{(1)}(x)\mathbf{e}_i\right)\mathbf{v}_u. \end{aligned}$$

Let us denote  $\mathbf{h} = \sum_{i=1}^{n'} R^{-(1-\lambda_i)q}\varphi_i^{(1)}(x)\mathbf{e}_i$  as before. Then by our previous argument, we have that

$$\begin{aligned} g_{\mathbf{r}}(q)U(\varphi(x_1))\mathbf{a}_u &= U(rR^{2l}\mathbf{h} + rR^{2l}\sum_{i=n'+1}^n R^{-(1-\lambda_i)q}\varphi_i^{(1)}(x)\mathbf{e}_i)\mathbf{v}_u \\ &= (a_{u,+} + rR^{2l}a_{u,1} + rR^{2l}\sum_{i=n'+1}^n R^{-(1-\lambda_i)q}\varphi_i^{(1)}(x)a_{u,i})\mathbf{w}_+ \\ &\quad + \sum_{i=1}^{n'} a_{u,i}z(\mathfrak{k})\mathbf{w}_i + \sum_{i=n'+1}^n a_{u,i}\mathbf{w}_i. \end{aligned}$$

Since  $|a_{u,1}| \leq CR^{-l}$ , and since for  $i = n' + 1, \dots, n$ ,  $(1 - \lambda_i)q > l$ ,  $|a_{u,i}| < C$ , and  $|\varphi_i^{(1)}(x)| \leq C_1$ , we have that

$$\begin{aligned}
& |a_{u,+} + rR^{2l}a_{u,1} + rR^{2l} \sum_{i=n'+1}^n R^{-(1-\lambda_i)q} \varphi^{(1)}(x) a_{u,i}| \\
& \leq |a_{u,+}| + |r|R^{2l}|a_{u,1}| + |r|R^{2l} \sum_{i=n'+1}^n R^{-(1-\lambda_i)q} |\varphi^{(1)}(x)| |a_{u,i}| \\
& \leq C + R^{2l}CR^{-l} + R^{2l} \sum_{i=n'+1}^n R^{-l} C_1 C \\
& \leq C + R^{2l}CR^{-l} + R^{2l}nR^{-l}C_1C \\
& \leq C_2R^l
\end{aligned}$$

where  $C_2 = 2C + nC_1C > 0$ . This implies that for any  $u = 1, \dots, L$ , we have that

$$g_{\mathbf{r}}(q)U(\varphi(x_1))\mathbf{a}_u \in z(\mathfrak{k})([-C_2R^l, C_2R^l] \times [-CR^{-l}, CR^{-l}] \times [-C, C]^{n-1}).$$

Let us consider the range of  $g_{\mathbf{r}}(q-l)U(\varphi(x_1))\mathbf{a}_u = g_{\mathbf{r}}(-l)g_{\mathbf{r}}(q)U(\varphi(x_1))\mathbf{a}_u$ . Let us write  $g_{\mathbf{r}}(-l) = d_2(l)d_1(l)$  where

$$d_1(l) = \begin{bmatrix} b^{-l} & & & & \\ & b^{r_1l} \mathbf{I}_{n_1} & & & \\ & & b^{r_{n'+1}l} & & \\ & & & \ddots & \\ & & & & b^{r_nl} \end{bmatrix},$$

and

$$d_2(l) = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & b^{-(r_1-r_2)l} & & \\ & & & \ddots & \\ & & & & b^{-(r_1-r_{n'})l} \\ & & & & & \mathbf{I}_{n-n'} \end{bmatrix}.$$

Then we have that

$$g_{\mathbf{r}}(q-l)U(\varphi(x_1))\mathbf{a}_u \in d_2(l)d_1(l)z(\mathfrak{k})([-C_2R^l, C_2R^l] \times [-CR^{-l}, CR^{-l}] \times [-C, C]^{n-1}).$$

By the definition of  $z(\mathfrak{k})$ , we have that  $d_1(l)z(\mathfrak{k}) = z(\mathfrak{k})d_1(l)$ . Therefore, we have that

$$\begin{aligned}
& d_1(l)z(\mathfrak{k})([-C_2R^l, C_2R^l] \times [-CR^{-l}, CR^{-l}] \times [-C, C]^{n-1}) \\
& = z(\mathfrak{k})d_1(l)([-C_2R^l, C_2R^l] \times [-CR^{-l}, CR^{-l}] \times [-C, C]^{n-1}) \\
& = z(\mathfrak{k})([-C_2b^{r_1l}, C_2b^{r_1l}] \times [-Cb^{-l}, Cb^{-l}] \times [-Cb^{r_1l}, Cb^{r_1l}]^{n_1-1} \times \prod_{i=n'+1}^n [-Cb^{r_{i1}l}, Cb^{r_{i1}l}]) \\
& \subset z(\mathfrak{k})([-C_2b^{r_1l}, C_2b^{r_1l}] \times [-1, 1] \times [-Cb^{r_1l}, Cb^{r_1l}]^{n'-1} \times \prod_{i=n'+1}^n [-Cb^{r_{i1}l}, Cb^{r_{i1}l}]).
\end{aligned}$$

It is easy to see that

$$z(\mathfrak{k})([-C_2b^{r_1l}, C_2b^{r_1l}] \times [-1, 1] \times [-Cb^{r_1l}, Cb^{r_1l}]^{n'-1} \times \prod_{i=n'+1}^n [-Cb^{r_{i1}l}, Cb^{r_{i1}l}])$$

can be covered by a collection  $\mathcal{B}$  of  $O(b^\lambda)$  balls of radius 1 where  $\lambda = n'r_1 + \sum_{i=n'+1}^n r_i$ . Then we have that

$$\begin{aligned}
g_{\mathbf{r}}(q-l)U(\varphi(x_1))\mathbf{a}_u & \in d_2(l) \bigcup_{B \in \mathcal{B}} B \\
& = \bigcup_{B \in \mathcal{B}} d_2(l)B.
\end{aligned}$$



Since  $d_2(l)$  is a contracting map, for every  $B \in \mathcal{B}$ , there exists a ball  $B'$  of radius  $C$  such that  $d_2(l)B \subset B'$ . Let  $\mathcal{B}'$  denote the collection of all such  $B'$ 's. Then we have that

$$g_{\mathbf{r}}(q-l)U(\varphi(x_1))\mathbf{a}_u \in \bigcup_{B' \in \mathcal{B}'} B'.$$

Since  $g_{\mathbf{r}}(q-l)U(\varphi(x_1))\mathbf{a}_u \in g_{\mathbf{r}}(q-l)U(\varphi(x_1))\mathbb{Z}^{n+1}$ , we have that

$$g_{\mathbf{r}}(q-l)U(\varphi(x_1))\mathbf{a}_u \in \bigcup_{B' \in \mathcal{B}'} B' \cap \Lambda,$$

where  $\Lambda = g_{\mathbf{r}}(q-l)U(\varphi(x_1))\mathbb{Z}^{n+1}$ . By our assumption,  $x_1 \in I_{q-1,1}$  for some  $I_{q-1,1} \in \mathcal{I}_{q-1}$ . This implies that  $x_1 \in I_{q-l}$  for some  $I_{q-l} \in \mathcal{I}_{q-l}$ . Therefore,  $\Lambda = g_{\mathbf{r}}(q-l)U(\varphi(x_1))\mathbb{Z}^{n+1} \in K_{\kappa}$ , i.e.,  $\Lambda$  does not contain any nonzero vectors with norm  $\leq \kappa$ . Therefore, there exists a constant  $C_4$  such that every ball of radius 1 contains at most  $C_4\kappa^{-n-1} = C_4R^{(n+1)m}$  points in  $\Lambda$ . Thus, we have that

$$\begin{aligned} |\mathcal{D}_{q,l}(I_p)| &= |\{g_{\mathbf{r}}(q-l)U(\varphi(x_1))\mathbf{a}_u : 1 \leq u \leq L\}| \leq \sum_{B' \in \mathcal{B}'} |B' \cap \Lambda| \\ &\leq \sum_{B' \in \mathcal{B}'} C_4 R^{(n+1)m} \\ &\leq C_5 b^{\lambda l + 4nm} \leq C_5 b^{(\lambda + \frac{1}{200n})l}, \end{aligned}$$

where  $C_5 = C_3 C_4$  and  $\lambda = n'r_1 + \sum_{i=n'+1}^n r_i$ . Now let us estimate  $\lambda$ . In fact,

$$\begin{aligned} \lambda &= \sum_{i=1}^n r_i + \sum_{i=1}^{n'} (r_1 - r_i) \\ &= 1 + \sum_{i=1}^{n'} (r_1 - r_i). \end{aligned}$$

By our assumption, for  $i = 1, \dots, n'$ , we have that  $r_1 - r_i \leq \frac{l}{q} \leq \frac{1}{100n^2}$ . Therefore, we have that

$$\lambda \leq 1 + n \frac{1}{100n^2} = 1 + \frac{1}{100n}.$$

Thus, we have that

$$|\mathcal{D}_{q,l}(I_p)| \leq C_5 b^{(1 + \frac{1}{100n} + \frac{1}{200n})l} \leq C_5 R^{(1 - \frac{1}{10n})l}.$$

The last inequality above holds because  $b = R^{\frac{1}{1+r_1}} \leq R^{\frac{n}{n+1}}$ .

This completes the proof. □

## 5. PROOF OF THE MAIN RESULT

In this section we will finish the proof of Proposition 3.7. By our discussion in §1 and §3, Proposition 3.7 implies Theorem 3.5 and Theorem 1.7.

The structure of the section is as follows. In the first subsection, we will prove Proposition 3.7 for the case  $q \leq 10^6 n^4 Nm$ . The second, third and fourth subsections are devoted to the proof for the case  $q > 10^6 n^4 Nm$ . The key point is to estimate  $F(\hat{\mathcal{I}}_{q,p}, I_p)$  for  $I_p \in \mathcal{I}_p$ . The second subsection deals with the case  $p = q - 4000n^2 Nm$ . The third subsection deals with the case  $p = q - 2l$  where  $2000n^2 Nm < l < 2n'q$ . The fourth subsection deals with the case  $p = 0$ .

The third and fourth subsections contain some technical results on the canonical representation of  $\mathrm{SL}(n+1, \mathbb{R})$  on  $\bigwedge^i V$  for  $i = 2, \dots, n$ . They are also main technical contributions of this paper.

Our basic tool is the following non-divergence theorem due to Kleinbock and Margulis:

**Theorem 5.1** (see [KM98, Theorem 5.2]). *There exist constants  $C, \alpha > 0$  such that the following holds: For any subinterval  $J \subset I$  and any  $t \geq 0$ , if for any  $i = 1, 2, \dots, n$  and any  $\mathbf{v} = \mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_i \in \bigwedge^i \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$ ,*

$$\max\{\|g_{\mathbf{r}}(t)U(\varphi(x))\mathbf{v}\| : x \in J\} \geq \rho^i,$$

*then for any  $\epsilon > 0$ ,*

$$m(\{x : g_{\mathbf{r}}(t)U(\varphi(x))\mathbb{Z}^{n+1} \notin K_{\epsilon}\}) \leq C \left(\frac{\epsilon}{\rho}\right)^{\alpha} m(J).$$

*Remark 5.2.* The original version of Theorem 5.1 is more general. The version above is tailored for our setting. For the case where  $\varphi(x)$  is a polynomial, the statement is proved by Dani [Dan84].

We will also need the following result due to Nimish Shah [Sha10] on  $\mathrm{SL}(n+1, \mathbb{R})$  representations.

**Theorem 5.3** (see [Sha10, Proposition 4.9]). *Let  $\mathcal{V}$  be any finite dimensional representation of  $\mathrm{SL}(n+1, \mathbb{R})$  with a norm  $\|\cdot\|$  and let  $\mathbf{r}$  be any  $n$ -dimensional weight. There exists a constant  $c > 0$  such that for any nonzero vector  $v \in \mathcal{V}$  and any  $t \geq 0$ ,*

$$\max\{\|g_{\mathbf{r}}(t)U(\varphi(x))v\| : x \in I\} \geq c\|v\|.$$

*Remark 5.4.* The exact statement in [Sha10, Proposition 4.9] is different from the above version, but it easily implies the above statement.

The proof makes use of the fact that  $\varphi$  is not contained in any proper affine subspace in  $\mathbb{R}^n$ .

We may choose  $0 < \rho < 1$  small enough such that  $\rho^{n+1} < c$  where  $c$  denotes the constant from Theorem 5.3.

**5.1. The case where  $q$  is small.** In this subsection, let us assume that  $q \leq 10^6 n^4 N m$ . Then only  $\hat{\mathcal{I}}_{q,0} = \hat{I}_q$  is nonempty.

**Proposition 5.5.**

$$F(\hat{\mathcal{I}}_{q,0}, I) \ll R^{q-\alpha m}.$$

*Proof.* By Theorem 5.3 and our assumption on  $\rho$ , we have that for any  $i = 1, 2, \dots, n$  and  $\mathbf{v} = \mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_i \in \bigwedge^i \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$ ,

$$\max\{\|g_{\mathbf{r}}(q)U(\varphi(x))\mathbf{v}\| : x \in I\} \geq c\|\mathbf{v}\| \geq c \geq \rho^i.$$

Then by Theorem 5.1, we have that

$$m(\{x \in I : g_{\mathbf{r}}(q)U(\varphi(x))\mathbb{Z}^{n+1} \notin K_{2\kappa}\}) \leq C \left(\frac{2\kappa}{\rho}\right)^{\alpha}.$$

On the other hand, it is easy to see that  $g_{\mathbf{r}}(q)U(\varphi(I_q))\mathbb{Z}^{n+1} \subset X \setminus K_{2\kappa}$  for any  $I_q \in \hat{\mathcal{I}}_q$ . Therefore, we have that

$$\begin{aligned} F(\hat{\mathcal{I}}_{q,0}, I) R^{-q} &= m\left(\bigcup_{I_q \in \hat{\mathcal{I}}_q} I_q\right) \\ &= m(\{x \in I : g_{\mathbf{r}}(q)U(\varphi(x))\mathbb{Z}^{n+1} \notin K_{2\kappa}\}) \leq C_6 \kappa^{\alpha} = C_6 R^{-\alpha m} \end{aligned}$$

where  $C_6 = C \left(\frac{2}{\rho}\right)^{\alpha}$ . This finishes the proof.  $\square$

Let us choose  $M > 1$  such that  $M^\alpha > 1000^{10^6 n^4 N}$ .

*Proof of Proposition 3.7 for  $q \leq 10^6 n^4 Nm$ .* It suffices to show that

$$\left(\frac{4}{R}\right)^q F(\hat{\mathcal{I}}_{q,0}, I)$$

can be arbitrarily small. In fact, by Proposition 5.5, we have that

$$\begin{aligned} \left(\frac{4}{R}\right)^q F(\hat{\mathcal{I}}_{q,0}, I) &= \left(\frac{4}{R}\right)^q O(R^{q-\alpha m}) \\ &= O\left(\frac{4^q}{R^{\alpha m}}\right) = O\left(\frac{4^{10^6 n^4 Nm}}{R^{\alpha m}}\right) = O\left(\left(\frac{4}{1000}\right)^{10^6 n^4 Nm}\right). \end{aligned}$$

Then it is easy to see that  $\left(\frac{4}{R}\right)^q F(\hat{\mathcal{I}}_{q,0}, I) \rightarrow 0$  as  $m \rightarrow \infty$ .

This completes the proof for  $q \leq 10^6 n^4 Nm$ .  $\square$

**5.2. The generic case.** The rest of the section is devoted to the proof of Proposition 3.7 for  $q > 10^6 n^4 Nm$ . In the following subsections, we will estimate  $F(\hat{\mathcal{I}}_{q,p}, I_p)$  for different  $p$ 's. In this subsection we will estimate  $F(\hat{\mathcal{I}}_{q,p}, I_p)$  for  $p = q - 4000n^2 Nm$ . We call it the generic case.

**Proposition 5.6.** *Let  $q > 10^6 n^4 Nm$  and  $p = q - 4000n^2 Nm$ . Then for any  $I_p \in \mathcal{I}_p$ , we have that*

$$F(\hat{\mathcal{I}}_{q,p}, I_p) \ll R^{q-p-\alpha m}.$$

*Proof.* Let us fix  $I_p \in \mathcal{I}_p$ . Let  $I_p = [a, b]$  and  $I'_p = [a - R^{-q+2000n^2 Nm}, b + R^{-q+2000n^2 Nm}]$ . It is easy to see that  $I_p \subset I'_p$  and  $m(I'_p) < 2m(I_p)$ .

If  $F(\hat{\mathcal{I}}_{q,p}, I_p) = 0$ , then the statement apparently holds.

Suppose  $F(\hat{\mathcal{I}}_{q,p}, I_p) > 0$ , let us take  $I_q \in \hat{\mathcal{I}}_{q,p}$  and  $x \in I_q \cap I_p$ . Then for any  $i = 1, \dots, n$  and  $\mathbf{v} = \mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_i \in \bigwedge^i \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$ , we have that

$$\max\{\|g_{\mathbf{r}}(q)U(\varphi(x'))\mathbf{v}\| : x' \in [x - R^{-q+2000n^2 Nm}, x + R^{-q+2000n^2 Nm}]\} \geq \rho^i.$$

It is easy to see that  $[x - R^{-q+2000n^2 Nm}, x + R^{-q+2000n^2 Nm}] \subset I'_p$ . Therefore, we have that

$$\max\{\|g_{\mathbf{r}}(q)U(\varphi(x'))\mathbf{v}\| : x' \in I'_p\} \geq \rho^i.$$

By Theorem 5.1, we have that

$$m(\{x \in I'_p : g_{\mathbf{r}}(q)U(\varphi(x))\mathbb{Z}^{n+1} \notin K_{2\kappa}\}) \leq C \left(\frac{2\kappa}{\rho}\right)^\alpha m(I'_p).$$

This implies that

$$m(\{x \in I_p : g_{\mathbf{r}}(q)U(\varphi(x))\mathbb{Z}^{n+1} \notin K_{2\kappa}\}) \leq C \left(\frac{2\kappa}{\rho}\right)^\alpha m(I'_p) \leq 2C \left(\frac{2\kappa}{\rho}\right)^\alpha m(I_p).$$

On the other hand, it is easy to see that  $g_{\mathbf{r}}(q)U(\varphi(I_q))\mathbb{Z}^{n+1} \subset X \setminus K_{2\kappa}$  for any  $I_q \in \hat{\mathcal{I}}_q$ . Therefore we have that

$$\begin{aligned} & F(\hat{\mathcal{I}}_{q,p}, I_p) R^{-q} \\ & \leq m(\{x \in I_p : g_{\mathbf{r}}(q)U(\varphi(x))\mathbb{Z}^{n+1} \notin K_{2\kappa}\}) \\ & \leq 2C \left(\frac{2\kappa}{\rho}\right)^\alpha m(I_p) \\ & = 2C \left(\frac{2}{\rho}\right)^\alpha \kappa^\alpha R^{-p} = C_7 R^{-p-\kappa m} \end{aligned}$$

where  $C_7 = 2C \left(\frac{2}{\rho}\right)^\alpha$ . This proves the statement.  $\square$

By Proposition 5.6, we have that for  $p = q - 4000n^2Nm$  and any  $I_p \in \mathcal{I}_p$ , the following holds:

$$(5.1) \quad \begin{aligned} \left(\frac{4}{R}\right)^{q-p} F(\hat{\mathcal{I}}_{q,p}, I_p) &\ll \left(\frac{4}{R}\right)^{q-p} R^{q-p-\alpha m} \\ &= \frac{4^{4000n^2Nm}}{R^{\alpha m}} = \left(\frac{4}{1000}\right)^{4000n^2Nm}. \end{aligned}$$

Then it is easy to see that  $\left(\frac{4}{R}\right)^{q-p} F(\hat{\mathcal{I}}_{q,p}, I_p) \rightarrow 0$  as  $m \rightarrow \infty$ .

**5.3. Dangerous case.** In this subsection, we will consider the case where  $2000n^2Nm < l < 2\eta'q$  and  $p = q - 2l$ . We call this case the  $(q, l)$ -dangerous case.

**Proposition 5.7.** *For any  $I_p \in \mathcal{I}_p$ , we have that*

$$F(\hat{\mathcal{I}}_{q,p}, I_p) \ll R^{q-p-\frac{l}{20n}}.$$

Let us recall that for  $1000n^2Nm < l' < \eta'q$ , a  $(q, l')$ -dangerous interval  $\Delta_{q,l'}(\mathbf{a})$  associated with a nonzero integer vector  $\mathbf{a} \in \mathbb{Z}^{n+1}$  is a closed interval of the form

$$\Delta_{q,l'}(\mathbf{a}) = [x - R^{-q+l'}, x + R^{-q+l'}]$$

such that  $I_q \subset \Delta_{q,l'}(\mathbf{a})$  for some  $I_q \in \hat{\mathcal{I}}_q$  and

$$\max\{\|g_{\mathbf{r}}(q)U(\varphi(x'))\mathbf{a}\| : x' \in \Delta_{q,l'}(\mathbf{a})\} = c\rho$$

for some  $c \in [1/2, 1]$ .

The following lemma is crucial to prove Proposition 5.7 and is one of the main technical contributions of this paper:

**Lemma 5.8.** *For any  $i = 1, \dots, n$  and  $I_q \in \hat{\mathcal{I}}_{q,p}(i)$  intersecting  $I_p$ , either there exists a  $(q, l')$ -dangerous interval  $\Delta_{q,l'}(\mathbf{a})$  containing  $I_q$  for some  $l/2 \leq l' \leq l$ , or there exists  $x \in I_q$  and*

$$\mathbf{v} = \mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_i \in \bigwedge^i \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$$

such that if we write

$$g_{\mathbf{r}}(q)U(\varphi(x))\mathbf{v} = \mathbf{w}_+ \wedge \mathbf{w}^{(i-1)} + \mathbf{w}^{(i)}$$

where  $\mathbf{w}^{(i-1)} \in \bigwedge^{i-1} W$  and  $\mathbf{w}^{(i)} \in \bigwedge^i W$ , then we have that  $\|\mathbf{w}_+ \wedge \mathbf{w}^{(i-1)}\| = \|\mathbf{w}^{(i-1)}\| \leq \rho^i$  and  $\|\mathbf{w}^{(i)}\| \leq \rho^i R^{-l/2}$ .

*Proof.* If  $i = 1$ , then the first statement apparently holds. We may assume that  $i \geq 2$ .

By the definition of  $\hat{\mathcal{I}}_{q,p}(i)$ , there exists  $\mathbf{v} = \mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_i \in \bigwedge^i \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$  such that for any  $x \in I_q$ ,

$$\max\{\|g_{\mathbf{r}}(q)U(\varphi(x'))\mathbf{v}\| : x' \in [x - R^{-q+l}, x + R^{-q+l}]\} = c\rho^i$$

for some  $c \in [1/2, 1]$ .

Without loss of generality, we may assume that the sublattice  $L_i$  generated by  $\{\mathbf{v}_1, \dots, \mathbf{v}_i\}$  is a primitive  $i$ -dimensional sublattice of  $\mathbb{Z}^{n+1}$ . Then  $\Lambda_i = g_{\mathbf{r}}(q)U(\varphi(x))L_i$  is a primitive  $i$ -dimensional sublattice of  $\Lambda = g_{\mathbf{r}}(q)U(\varphi(x))\mathbb{Z}^{n+1}$ . For simplicity, let us denote  $g = g_{\mathbf{r}}(q)U(\varphi(x))$ . Let us choose the Minkowski reduced basis  $\{g\mathbf{v}'_1, \dots, g\mathbf{v}'_i\}$  of  $\Lambda_i$ . Since

$$d(\Lambda_i) = \|g\mathbf{v}\| \leq \rho^i,$$

we have that  $\|g\mathbf{v}'_1\| \leq \rho$  by the Minkowski Theorem.

Let us follow the argument in the proof of Proposition 4.1. Recall that for  $j = 1, \dots, n$ ,  $\lambda_j = \frac{1+r_j}{1+r_1}$ . Let  $1 \leq n' \leq n$  be the largest index  $j$  such that  $(1 - \lambda_j)q \leq l$ . Let us write

$$\varphi(s) = (\varphi_1(s), \dots, \varphi_n(s))^T.$$

By Standing Assumption **A.2**, we have that  $c_1 \leq |\varphi_i^{(1)}(s)| \leq C_1$  for any  $i = 1, \dots, n$  and  $s \in I$ . Let  $\mathbf{h} = \sum_{i=1}^{n'} R^{-(1-\lambda_i)q} \varphi^{(1)}(s) \mathbf{e}_i$ . For any  $x' \in [x - R^{-q+l}, x + R^{-q+l}]$ , let us write  $x' = x + rR^{-q+l}$  where  $r \in [-1, 1]$ . By the same argument as in the proof of Proposition 4.1, we have that

$$g_{\mathbf{r}}(q)U(\varphi(x')) = U(O(1))U(rR^l \mathbf{h})g_{\mathbf{r}}(q)U(\varphi(x)) = U(O(1))U(rR^l \mathbf{h})g.$$

Therefore, we have that

$$\|U(rR^l \mathbf{h})g\mathbf{v}\| \leq \rho^i$$

for any  $r \in [-1, 1]$ .

Following the notation in the proof of Proposition 4.1, let us denote  $\mathbf{h} = \mathfrak{k} \cdot \mathbf{e}_1$  for  $\mathfrak{k} \in \text{SO}(n)$  and

$$z(\mathfrak{k}) = \begin{bmatrix} 1 & \\ & \mathfrak{k} \end{bmatrix} \in Z.$$

For  $j = 1, \dots, i$ , let us write

$$g\mathbf{v}'_j = a_+(j)\mathbf{w}_+ + a_1(j)z(\mathfrak{k})\mathbf{w}_1 + \mathbf{w}'(j)$$

where  $\mathbf{w}'(j) \in z(\mathfrak{k})W_2$ . Then

$$\begin{aligned} g\mathbf{v} &= (g\mathbf{v}'_1) \wedge \dots \wedge (g\mathbf{v}'_i) \\ &= \bigwedge_{j=1}^i (a_+(j)\mathbf{w}_+ + a_1(j)z(\mathfrak{k})\mathbf{w}_1 + \mathbf{w}'(j)) \\ &= \mathbf{w}_+ \wedge (z(\mathfrak{k})\mathbf{w}_1) \wedge \left( \sum_{j < j'} \epsilon_{+,1}(j, j') a_+(j) a_1(j') \bigwedge_{k \neq j, j'} \mathbf{w}'(k) \right) \\ &\quad + \mathbf{w}_+ \wedge \left( \sum_{j=1}^i \epsilon_+(j) a_+(j) \bigwedge_{k \neq j} \mathbf{w}'(k) \right) + (z(\mathfrak{k})\mathbf{w}_1) \wedge \left( \sum_{j=1}^i \epsilon_1(j) a_1(j) \bigwedge_{k \neq j} \mathbf{w}'(k) \right) \\ &\quad + \bigwedge_{j=1}^i \mathbf{w}'(j) \end{aligned}$$

where  $\epsilon_{+,1}(j, j'), \epsilon_+(j), \epsilon_1(j) \in \{\pm 1\}$  for every  $j, j' \in \{1, \dots, i\}$ . By our discussion in §2.2 on the representation of  $\text{SL}(2, \mathbf{h})$  on  $\bigwedge^i V$ , we have that

$$\begin{aligned} U(rR^l \mathbf{h})g\mathbf{v} &= \mathbf{w}_+ \wedge (z(\mathfrak{k})\mathbf{w}_1) \wedge \left( \sum_{j < j'} \epsilon_{+,1}(j, j') a_+(j) a_1(j') \bigwedge_{k \neq j, j'} \mathbf{w}'(k) \right) \\ &\quad + \mathbf{w}_+ \wedge \left( \sum_{j=1}^i \epsilon_+(j) a_+(j) \bigwedge_{k \neq j} \mathbf{w}'(k) \right) \\ &\quad + rR^l \mathbf{w}_+ \wedge \left( \sum_{j=1}^i \epsilon_1(j) a_1(j) \bigwedge_{k \neq j} \mathbf{w}'(k) \right) \\ &\quad + (z(\mathfrak{k})\mathbf{w}_1) \wedge \left( \sum_{j=1}^i \epsilon_1(j) a_1(j) \bigwedge_{k \neq j} \mathbf{w}'(k) \right) + \bigwedge_{j=1}^i \mathbf{w}'(j). \end{aligned}$$

Since  $\|U(rR^l \mathbf{h})g\mathbf{v}\| \leq \rho^i$  for any  $r \in [-1, 1]$ , we have that

$$\left\| \sum_{j=1}^i \epsilon_1(j) a_1(j) \bigwedge_{k \neq j} \mathbf{w}'(k) \right\| \leq \rho^i R^{-l}.$$

Let us consider the following two cases:

- (1)  $|a_1(1)| \leq R^{-l/2}$ .
- (2)  $|a_1(1)| > R^{-l/2}$ .

Let us first suppose  $|a_1(1)| \leq R^{-l/2}$ . Note that  $\|g\mathbf{v}'_1\| \leq \rho$ . Then by repeating the calculation in the proof of Proposition 4.1, we conclude that

$$\max\{\|g_{\mathbf{r}}(q)U(\varphi(x'))\mathbf{v}'_1\| : x' \in [x - R^{-q+l/2}, x + R^{-q+l/2}]\} \leq \rho.$$

On the other hand, by our definition on  $\hat{\mathcal{I}}_{q,p}(i)$ , we have that

$$\max\{\|g_{\mathbf{r}}(q)U(\varphi(x'))\mathbf{v}'_1\| : x' \in [x - R^{-q+l}, x + R^{-q+l}]\} \geq \frac{1}{2}\rho.$$

This implies that  $I_q \subset \Delta_{q,l'}(\mathbf{v}'_1)$  for some  $l/2 \leq l' \leq l$ . This proves the first part of the statement.

Now let us suppose  $|a_1(1)| > R^{-l/2}$ . Then we have that

$$\begin{aligned} \epsilon_1(1)a_1(1) \bigwedge_{j=1}^i \mathbf{w}'(j) &= \mathbf{w}'(1) \wedge \left( \epsilon_1(1)a_1(1) \bigwedge_{k \neq 1} \mathbf{w}'(k) \right) \\ &= \mathbf{w}'(1) \wedge \left( \sum_{j=1}^i \epsilon_1(j)a_1(j) \bigwedge_{k \neq j} \mathbf{w}'(k) \right). \end{aligned}$$

Therefore, we have that

$$\begin{aligned} |a_1(1)| \left\| \bigwedge_{j=1}^i \mathbf{w}'(j) \right\| &= \left\| \mathbf{w}'(1) \wedge \left( \sum_{j=1}^i \epsilon_1(j)a_1(j) \bigwedge_{k \neq j} \mathbf{w}'(k) \right) \right\| \\ &\leq \|\mathbf{w}'(1)\| \left\| \sum_{j=1}^i \epsilon_1(j)a_1(j) \bigwedge_{k \neq j} \mathbf{w}'(k) \right\| \\ &\leq \rho \cdot \rho^i R^{-l} = \rho^{i+1} R^{-l}. \end{aligned}$$

Since  $|a_1(1)| > R^{-l/2}$  and  $\rho < 1$ , we have that

$$\left\| \bigwedge_{j=1}^i \mathbf{w}'(j) \right\| \leq \rho^i R^{-l/2}.$$

If we write

$$g\mathbf{v} = \mathbf{w} \wedge \mathbf{w}^{(i-1)} + \mathbf{w}^{(i)}$$

where  $\mathbf{w}^{(i-1)} \in \bigwedge^{i-1} W$  and  $\mathbf{w}^{(i)} \in \bigwedge^i W$ , then

$$\mathbf{w}^{(i)} = (z(\mathfrak{k})\mathbf{w}_1) \wedge \left( \sum_{j=1}^i \epsilon_1(j)a_1(j) \bigwedge_{k \neq j} \mathbf{w}'(k) \right) + \bigwedge_{j=1}^i \mathbf{w}'(j).$$

By our previous argument, we have that

$$\|\mathbf{w}^{(i)}\| \leq \rho^i R^{-l/2}.$$

This proves the second part of the statement.  $\square$

The following lemma takes care of the second case of Lemma 5.8.

**Lemma 5.9.** *Let  $i \in \{2, \dots, n\}$ . Let  $\mathcal{D}_{q,p}(I_p, i)$  denote the collection of  $I_q \in \hat{\mathcal{I}}_{q,p}$  intersecting  $I_p$  and not contained in any  $(q, l')$ -dangerous interval for any  $l/2 \leq l' \leq l$ . Let*

$$D_{q,p}(I_p, i) := \bigcup_{I_q \in \mathcal{D}_{q,p}(I_p, i)} I_q.$$

*Then for any closed subinterval  $J \subset I_p$  of length  $R^{-q+(1+\frac{1}{2n})l}$ , we have that*

$$m(D_{q,p}(I_p, i) \cap J) \ll R^{-\frac{l}{20n}} m(J).$$

*Proof.* Let us fix a closed subinterval  $J \subset I_p$  of length  $R^{-q+(1+\frac{1}{2n})l}$ .

For any  $x \in I_q \in \mathcal{D}_{q,p}(I_p, i)$ , there exists  $\mathbf{v} = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_i \in \bigwedge^i \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$  such that

$$\max\{\|g_{\mathbf{r}}(q)U(\varphi(x'))\mathbf{v}\| : x' \in [x - R^{-q+l}, x + R^{-q+l}]\} \leq \rho^i.$$

Let us denote the interval  $[x - R^{-q+l}, x + R^{-q+l}]$  by  $\Delta_{q,l}(\mathbf{v}, i)$ . Then every  $I_q \in \mathcal{D}_{q,l}(I_p, i)$  is contained in some  $\Delta_{q,l}(\mathbf{v}, i)$  and every  $\Delta_{q,l}(\mathbf{v}, i)$  contains at most  $O(R^l)$  different  $I_q \in \mathcal{D}_{q,l}(I_p, i)$ .

We will follow the notation used in the proof of Lemma 5.8. Let  $g = g_{\mathbf{r}}(q)U(\varphi(x))$ ,  $\mathbf{h} = \mathbf{k} \cdot \mathbf{e}_1$  and

$$z(\mathbf{k}) = \begin{bmatrix} 1 & \\ & \mathbf{k} \end{bmatrix} \in Z$$

be as in the proof of Lemma 5.8. For  $j = 1, \dots, i$ , let us write

$$\begin{aligned} g\mathbf{v}_j &= a_+(j)\mathbf{w}_+ + a_1(j)z(\mathbf{k})\mathbf{w}_1 + \mathbf{w}'(j) \\ &= a_+(j)\mathbf{w}_+ + \mathbf{w}(j) \end{aligned}$$

where  $\mathbf{w}'(j) \in z(\mathbf{k})W_2$  and  $\mathbf{w}(j) = a_1(j)z(\mathbf{k})\mathbf{w}_1 + \mathbf{w}'(j) \in W$ . Then

$$\begin{aligned} g\mathbf{v} &= \mathbf{w}_+ \wedge (z(\mathbf{k})\mathbf{w}_1) \wedge \left( \sum_{j < j'} \epsilon_{+,1}(j, j') a_+(j) a_1(j') \bigwedge_{k \neq j, j'} \mathbf{w}'(k) \right) \\ &\quad + \mathbf{w}_+ \wedge \left( \sum_{j=1}^i \epsilon_+(j) a_+(j) \bigwedge_{k \neq j} \mathbf{w}'(k) \right) \\ &\quad + (z(\mathbf{k})\mathbf{w}_1) \wedge \left( \sum_{j=1}^i \epsilon_1(j) a_1(j) \bigwedge_{k \neq j} \mathbf{w}'(k) \right) + \bigwedge_{j=1}^i \mathbf{w}'(j). \end{aligned}$$

By Lemma 5.8, we have that

$$\left\| (z(\mathbf{k})\mathbf{w}_1) \wedge \left( \sum_{j=1}^i \epsilon_1(j) a_1(j) \bigwedge_{k \neq j} \mathbf{w}'(k) \right) \right\| \leq \rho^i R^{-l}$$

and

$$\left\| \bigwedge_{j=1}^i \mathbf{w}'(j) \right\| \leq \rho^i R^{-l/2}.$$

Let us take the collection of all possible  $\Delta_{q,l}(\mathbf{v}, i)$ 's intersecting  $J$ , say

$$\{\Delta_{q,l}(\mathbf{v}(M), i) = [x(M) - R^{-q+l}, x(M) + R^{-q+l}] : M = 1, \dots, L\}.$$

For simplicity, let us denote  $g(M) = g_{\mathbf{r}}(q)U(\varphi(x(M)))$  for  $M = 1, \dots, L$ . For  $M = 1, \dots, L$ , let us write

$$g(M)\mathbf{v}(M) = \mathbf{w}_+ \wedge \mathbf{w}^{(i-1)}(M) + (z(\mathbf{k})\mathbf{w}_1) \wedge (\mathbf{w}')^{(i-1)}(M) + \mathbf{w}^{(i)}(M)$$

where  $\mathbf{w}^{(i-1)}(M) \in \bigwedge^{i-1} W$ ,  $(\mathbf{w}')^{(i-1)}(M) \in \bigwedge^{i-1} z(\mathbf{k})W_2$  and  $\mathbf{w}^{(i)}(M) \in \bigwedge^i z(\mathbf{k})W_2$ . By our previous discussion, we have that

$$\|\mathbf{w}_+ \wedge \mathbf{w}^{(i-1)}(M)\| \leq \rho^i,$$

$$\|(\mathbf{w}')^{(i-1)}(M)\| = \|(z(\mathbf{k})\mathbf{w}_1) \wedge (\mathbf{w}')^{(i-1)}(M)\| \leq \rho^i R^{-l},$$

and

$$\|\mathbf{w}^{(i)}(M)\| \leq \rho^i R^{-l/2}.$$

Now let us consider  $g(1)\mathbf{v}(M)$ . Let us write  $x(1) - x(M) = rR^{-q+(1+\frac{1}{2n})l}$  where  $r \in [-1, 1]$ . By our previous discussion, we have that

$$\begin{aligned} g(1) &= g_{\mathbf{r}}(q)U(\varphi(x(1))) = U(O(1))U(rR^{(1+\frac{1}{2n})l}\mathbf{h})g_{\mathbf{r}}(q)U(\varphi(x(M))) \\ &= U(O(1))U(rR^{(1+\frac{1}{2n})l}\mathbf{h})g(M). \end{aligned}$$

Therefore, we have that

$$g(1)\mathbf{v}(M) = U(O(1))U(rR^{(1+\frac{1}{2n})l}\mathbf{h})g(M)\mathbf{v}(M).$$

It is easy to see that we can ignore the contribution of  $U(O(1))$  and identify  $g(1)\mathbf{v}(M)$  with  $U(rR^{(1+\frac{1}{2n})l}\mathbf{h})g(M)\mathbf{v}(M)$ . Then we have that

$$\begin{aligned} g(1)\mathbf{v}(M) &= U(rR^{(1+\frac{1}{2n})l}\mathbf{h})g(M)\mathbf{v}(M) \\ &= \mathbf{w}_+ \wedge \mathbf{w}^{(i-1)}(M) + rR^{(1+\frac{1}{2n})l}\mathbf{w}_+ \wedge (\mathbf{w}')^{(i-1)}(M) \\ &\quad + (z(\mathfrak{k})\mathbf{w}_1) \wedge (\mathbf{w}')^{(i-1)}(M) + \mathbf{w}^{(i)}(M). \end{aligned}$$

Now let us look at the range of

$$g_{\mathbf{r}}(-l/2)g(1)\mathbf{v}(M) = g_{\mathbf{r}}(q - l/2)U(\varphi(x(1)))\mathbf{v}(M).$$

It is easy to see that  $g_{\mathbf{r}}(-l/2)\mathbf{w}_+ = b^{-l/2}\mathbf{w}_+$ ,  $\|g_{\mathbf{r}}(-l/2)z(\mathfrak{k})\mathbf{w}_1\| \leq b^{r_1l/2}\|z(\mathfrak{k})\mathbf{w}_1\|$ ,

$$\|g_{\mathbf{r}}(-l/2)\mathbf{w}^{(i-1)}(M)\| \leq b^{l/2}\|\mathbf{w}^{(i-1)}(M)\|,$$

$$\|g_{\mathbf{r}}(-l/2)(\mathbf{w}')^{(i-1)}(M)\| \leq b^{(1-r_1)l/2}\|(\mathbf{w}')^{(i-1)}(M)\|,$$

and

$$\|g_{\mathbf{r}}(-l/2)\mathbf{w}^{(i)}(M)\| \leq b^{l/2}\|\mathbf{w}^{(i)}(M)\|.$$

Since

$$\begin{aligned} g_{\mathbf{r}}(-l/2)g(1)\mathbf{v}(M) &= b^{-l/2}\mathbf{w}_+ \wedge (g_{\mathbf{r}}(-l/2)\mathbf{w}^{(i-1)}(M)) \\ &\quad + rR^{(1+\frac{1}{2n})l}b^{-l/2}\mathbf{w}_+ \wedge (g_{\mathbf{r}}(-l/2)(\mathbf{w}')^{(i-1)}(M)) \\ &\quad + (g_{\mathbf{r}}(-l/2)z(\mathfrak{k})\mathbf{w}_1) \wedge (g_{\mathbf{r}}(-l/2)(\mathbf{w}')^{(i-1)}(M)) \\ &\quad + g_{\mathbf{r}}(-l/2)\mathbf{w}^{(i)}(M), \end{aligned}$$

we have that

$$\begin{aligned} \|g_{\mathbf{r}}(-l/2)g(1)\mathbf{v}(M)\| &\leq b^{-l/2}\|\mathbf{w}_+ \wedge (g_{\mathbf{r}}(-l/2)\mathbf{w}^{(i-1)}(M))\| \\ &\quad + R^{(1+\frac{1}{2n})l}b^{-l/2}\|\mathbf{w}_+ \wedge (g_{\mathbf{r}}(-l/2)(\mathbf{w}')^{(i-1)}(M))\| \\ &\quad + \|g_{\mathbf{r}}(-l/2)z(\mathfrak{k})\mathbf{w}_1\| \cdot \|g_{\mathbf{r}}(-l/2)(\mathbf{w}')^{(i-1)}(M)\| \\ &\quad + \|g_{\mathbf{r}}(-l/2)\mathbf{w}^{(i)}(M)\| \\ &\leq b^{-l/2}b^{l/2}\|\mathbf{w}^{(i-1)}(M)\| + R^{(1+\frac{1}{2n})l}b^{-l/2}b^{(1-r_1)l/2}\|(\mathbf{w}')^{(i-1)}(M)\| \\ &\quad + b^{r_1l/2}\|z(\mathfrak{k})\mathbf{w}_1\| \cdot b^{(1-r_1)l/2}\|(\mathbf{w}')^{(i-1)}(M)\| + b^{l/2}\|\mathbf{w}^{(i)}(M)\| \\ &\leq b^{-l/2}b^{l/2}\rho^i + R^{(1+\frac{1}{2n})l}b^{-l/2}b^{(1-r_1)l/2}\rho^i R^{-l} \\ &\quad + b^{r_1l/2}b^{(1-r_1)l/2}\rho^i R^{-l} + b^{l/2}\rho^i R^{-l/2} \\ &\leq \rho^i + \rho^i + \rho^i R^{-l/2} + \rho^i \leq 1. \end{aligned}$$

For  $M = 1, \dots, L$ , let  $\Lambda_i(\mathbf{v}(M))$  denote the  $i$ -dimensional primitive sublattice of  $\mathbb{Z}^{n+1}$  corresponding to  $\mathbf{v}(M)$ . We will apply Proposition 2.2 to estimate  $L$ . Thus, let us keep the notation used there. By the inequality above, we have that  $g_{\mathbf{r}}(-l/2)g(1)\Lambda_i(\mathbf{v}(M)) \in \mathcal{C}_i(g_{\mathbf{r}}(-l/2)g(1)\mathbb{Z}^{n+1}, 1)$  for every  $M = 1, \dots, L$ . On the other hand, since  $x(1) \in I_q \in \hat{\mathcal{I}}_q$ , we have that

$$g_{\mathbf{r}}(-l/2)g(1)\mathbb{Z}^{n+1} = g_{\mathbf{r}}(q - l/2)U(\varphi(x(1)))\mathbb{Z}^{n+1} \in K_{\kappa}.$$



By Proposition 2.2, we have that

$$L \leq |\mathcal{C}_i(g_{\mathbf{r}}(-l/2)g(1)\mathbb{Z}^{n+1}, 1)| \leq \kappa^{-N} = R^{Nm}.$$

Therefore, we have that

$$\begin{aligned} m(D_{q,p}(I_p, i) \cap J) &\ll LR^{-q+l} \leq R^{-q+l+Nm} \\ &\leq R^{-q+l+\frac{l}{100n}} \leq R^{-\frac{l}{20n}} R^{-q+(1+\frac{1}{2n})l} = R^{-\frac{l}{20n}} m(J). \end{aligned}$$

This completes the proof.  $\square$

Lemma 5.9 easily implies the following:

**Corollary 5.10.** *Let us keep the notation as above. Then*

$$m(D_{q,p}(I_p, i)) \ll R^{-\frac{l}{20n}} m(I_p).$$

*Proof.* The statement follows from Lemma 5.9 by dividing  $I_p$  into subintervals of length  $R^{-q+(1+\frac{1}{2n})l}$ .  $\square$

Now we are ready to prove Proposition 5.7.

*Proof of Proposition 5.7.* Let us fix  $I_p \in \mathcal{I}_p$ . For every  $l/2 \leq l' \leq l$ , let us denote by  $D_{q,l'}(I_p)$  denote the union of  $(q, l')$ -dangerous intervals intersecting  $I_p$ . By Proposition 4.1, we have that  $m(D_{q,l'}(I_p)) = O\left(R^{-\frac{l'}{10n}}\right) m(I_p)$ . Therefore, we have that

$$\begin{aligned} m\left(\bigcup_{l/2 \leq l' \leq l} D_{q,l'}(I_p)\right) &\leq \sum_{l/2 \leq l' \leq l} m(D_{q,l'}(I_p)) \\ &\ll \sum_{l/2 \leq l' \leq l} R^{-\frac{l'}{10n}} m(I_p) \\ &\ll R^{-\frac{l}{20n}} m(I_p). \end{aligned}$$

By Corollary 5.10, we have that

$$\begin{aligned} m\left(\bigcup_{i=2}^n D_{q,p}(I_p, i)\right) &\leq \sum_{i=2}^n m(D_{q,p}(I_p, i)) \\ &\ll \sum_{i=2}^n R^{-\frac{l}{20n}} m(I_p) \ll R^{-\frac{l}{20n}} m(I_p) \end{aligned}$$

By Lemma 5.8, we have that

$$I_q \subset \bigcup_{l/2 \leq l' \leq l} D_{q,l'}(I_p) \bigcup_{i=2}^n D_{q,p}(I_p, i)$$

for any  $I_q \in \hat{\mathcal{I}}_{q,p}$ . Therefore, we have that

$$\begin{aligned} F(\hat{\mathcal{I}}_{q,p}, I_p) R^{-q} &\leq m\left(\bigcup_{l/2 \leq l' \leq l} D_{q,l'}(I_p) \bigcup_{i=2}^n D_{q,p}(I_p, i)\right) \\ &\leq m\left(\bigcup_{l/2 \leq l' \leq l} D_{q,l'}(I_p)\right) + m\left(\bigcup_{i=2}^n D_{q,p}(I_p, i)\right) \\ &\ll R^{-\frac{l}{20n}} m(I_p) = R^{-p-\frac{l}{20n}}. \end{aligned}$$

This proves that

$$F(\hat{\mathcal{I}}_{q,p}, I_p) \ll R^{q-p-\frac{l}{20n}}.$$

$\square$

By Proposition 5.7, we have that

$$\begin{aligned}
(5.2) \quad & \sum_{l=2000n^2Nm}^{2\eta'q} \left(\frac{4}{R}\right)^{2l} \max_{I_{q-2l} \in \mathcal{I}_{q-2l}} F(\hat{\mathcal{I}}_{q,q-2l}, I_{q-2l}) \\
& \ll \sum_{l=2000n^2Nm}^{2\eta'q} \left(\frac{4}{R}\right)^{2l} R^{2l - \frac{l}{20n}} \\
& \leq \sum_{l=2000n^2Nm}^{2\eta'q} \left(\frac{16}{1000}\right)^l \ll \left(\frac{16}{1000}\right)^{2000n^2Nm}.
\end{aligned}$$

From this it is easy to see that

$$(5.3) \quad \sum_{l=2000n^2Nm}^{2\eta'q} \left(\frac{4}{R}\right)^{2l} \max_{I_{q-2l} \in \mathcal{I}_{q-2l}} F(\hat{\mathcal{I}}_{q,q-2l}, I_{q-2l}) \rightarrow 0$$

as  $m \rightarrow \infty$ .

**5.4. Extremely dangerous case.** In this subsection we will estimate  $F(\hat{\mathcal{I}}_{q,0}, I)$ . We call this case the extremely dangerous case.

**Proposition 5.11.** *There exists a constant  $\nu > 0$  such that for any  $q > 10^6 n^4 Nm$ , we have that*

$$F(\hat{\mathcal{I}}_{q,0}, I) \ll R^{(1-\nu)q}.$$

Similar to Lemma 5.8, we have the following:

**Lemma 5.12.** *For any  $i = 1, \dots, n$  and  $I_q \in \hat{\mathcal{I}}_{q,0}(i)$ , either there exists a  $q$ -extremely dangerous interval  $\Delta_q(\mathbf{a})$  such that  $I_q \in \Delta_q(\mathbf{a})$ , or there exists  $\mathbf{v} = \mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_i \in \bigwedge^i \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$  such that the following holds: for any  $x \in I_q$ , if we write*

$$g_{\mathbf{r}}(q)U(\varphi(x))\mathbf{v} = \mathbf{w}_+ \wedge \mathbf{w}^{(i-1)} + \mathbf{w}^{(i)}$$

*where  $\mathbf{w}^{(i-1)} \in \bigwedge^{i-1} W$  and  $\mathbf{w}^{(i)} \in \bigwedge^i W$ , then  $\|\mathbf{w}_+ \wedge \mathbf{w}^{(i-1)}\| \leq \rho^i$  and  $\|\mathbf{w}^{(i)}\| \leq \rho^i R^{-\eta'q}$ .*

*Proof.* The proof is the same as the proof of Lemma 5.8.  $\square$

**Definition 5.13.** For  $i = 2, \dots, n$ , let  $\mathcal{D}_q(i)$  denote the collection of  $I_q \in \hat{\mathcal{I}}_{q,0}(i)$  such that the second case in Lemma 5.12 holds and let

$$D_q(i) := \bigcup_{I_q \in \mathcal{D}_q(i)} I_q.$$

Moreover, for  $I_q \in \mathcal{D}_q(i)$ , let  $\mathbf{v} = \mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_i \in \bigwedge^i \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$  be the vector given in the second case of Lemma 5.12. Then for  $x \in I_q$ , we can write

$$g_{\mathbf{r}}(q)U(\varphi(x))\mathbf{v} = \mathbf{w}_+ \wedge \mathbf{w}^{(i-1)} + \mathbf{w}^{(i)}$$

as in the second case of Lemma 5.12. For  $l \geq \eta'q$ , let  $\mathcal{D}'_{q,l}(i)$  denote the collection of  $I_q \in \mathcal{D}_q(i)$  such that

$$\rho^i R^{-l+1} \leq \|\mathbf{w}^{(i)}\| \leq \rho^i R^{-l},$$

and let

$$D'_{q,l}(i) := \bigcup_{I_q \in \mathcal{D}'_{q,l}(i)} I_q.$$

**Lemma 5.14.** *There exists a constant  $\nu > 0$  such that for any  $q > 10^6 n^4 Nm$  and any  $i = 2, \dots, n$ , we have that*

$$m(D_q(i)) \ll R^{-\nu q}.$$

*Proof.* For any  $\eta'q \leq l \leq 2\eta'q$ , using the same argument as in the proof of Lemma 5.9, we can prove that

$$m(D'_{q,l}(i)) \ll R^{-\frac{l}{20n}}.$$

Therefore, we have that

$$\begin{aligned} m\left(\bigcup_{l=\eta'q}^{2\eta'q} D'_{q,l}(i)\right) &\leq \sum_{l=\eta'q}^{2\eta'q} m(D'_{q,l}(i)) \\ &\ll \sum_{l=\eta'q}^{2\eta'q} R^{-\frac{l}{20n}} \ll R^{-\frac{\eta'q}{20n}}. \end{aligned}$$

Let us denote

$$\mathcal{D}'_q(i) := \bigcup_{l > 2\eta'q} \mathcal{D}'_{q,l}$$

and

$$D'_q(i) := \bigcup_{I_q \in \mathcal{D}'_q(i)} I_q.$$

Then it is enough to show that

$$m(D'_q(i)) \ll R^{-\nu q}.$$

For any  $I_q \in \mathcal{D}'_q(i)$  and  $x \in I_q$ , there exists  $\mathbf{v} = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_i \in \bigwedge^i \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$  such that if we write

$$g_{\mathbf{r}}(q)U(\varphi(x))\mathbf{v} = \mathbf{w}_+ \wedge \mathbf{w}^{(i-1)} + \mathbf{w}^{(i)}$$

where  $\mathbf{w}^{(i-1)} \in \bigwedge^{i-1} W$  and  $\mathbf{w}^{(i)} \in \bigwedge^i W$ , then we have that  $\|\mathbf{w}_+ \wedge \mathbf{w}^{(i-1)}\| \leq \rho^i$  and  $\|\mathbf{w}^{(i)}\| \leq \rho^i R^{-2\eta'q}$ .

Recall that  $\eta = (1 + r_1)\eta'$ . Let us deal with the following two cases separately:

- (1)  $r_n \geq \frac{\eta}{n}$ .
- (2) There exists  $1 < n_1 \leq n$  such that for  $r_i \geq \frac{\eta}{n}$  for  $1 \leq i < n_1$  and  $r_i < \frac{\eta}{n}$  for  $n_1 \leq i \leq n$ .

Let us first deal with the first case. For this case, let us define

$$g^\eta(t) := \begin{bmatrix} b^{-\eta t} & \\ & b^{\eta t/n} \mathbf{I}_n \end{bmatrix} \in \mathrm{SL}(n+1, \mathbb{R})$$

and  $g_{\mathbf{r},\eta}(t) := g^\eta(t)g_{\mathbf{r}}(t)$ . It is easy to see that

$$g^\eta(t)\mathbf{w}_+ = b^{-\eta t}\mathbf{w}_+ = R^{-\eta' t}\mathbf{w}_+,$$

and

$$g^\eta(t)\mathbf{w} = b^{\eta t/n}\mathbf{w} = R^{\eta' t/n}\mathbf{w}$$

for any  $\mathbf{w} \in W$ .

Then we have that

$$\begin{aligned} \|g_{\mathbf{r},\eta}(q)U(\varphi(x))\mathbf{v}\| &= \|g^\eta(q)(\mathbf{w}_+ \wedge \mathbf{w}^{(i-1)} + \mathbf{w}^{(i)})\| \\ &\leq \|g^\eta(q)(\mathbf{w}_+ \wedge \mathbf{w}^{(i-1)})\| + \|g^\eta(q)\mathbf{w}^{(i)}\| \\ &= b^{-\eta q(1-\frac{i-1}{n})}\|\mathbf{w}_+ \wedge \mathbf{w}^{(i-1)}\| + b^{\frac{\eta q i}{n}}\|\mathbf{w}^{(i)}\| \\ &\leq b^{-\frac{\eta q}{n}}\rho^i + b^{\eta q}R^{-2\eta'q}\rho^i \leq R^{-\frac{\eta'q}{n}}\rho^i. \end{aligned}$$

By the Minkowski Theorem, the above inequality implies that the lattice  $g_{\mathbf{r},\eta}(q)U(\varphi(x))\mathbb{Z}^{n+1}$  contains a nonzero vector with norm  $\leq R^{-\frac{\eta'q}{n^2}}\rho$ . Therefore, for any  $I_q \in \mathcal{D}'_q(i)$  we have that

$$g_{\mathbf{r},\eta}(q)U(\varphi(I_q))\mathbb{Z}^{n+1} \notin K_\sigma$$

where  $\sigma = R^{-\frac{\eta'q}{n^2}}\rho$ . By Theorem 5.3, for any  $j = 1, \dots, n$  and any  $\mathbf{v} = \mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_j \in \bigwedge^j \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$ , we have that

$$\max\{g_{\mathbf{r},\eta}(q)U(\varphi(x))\mathbf{v} : x \in I\} \geq \rho^i.$$

Then by Theorem 5.1, we have that

$$m(\{x \in I : g_{\mathbf{r},\eta}(q)U(\varphi(x))\mathbb{Z}^{n+1} \notin K_\sigma\}) \ll \sigma^\alpha = R^{-\frac{\alpha\eta'q}{n^2}}.$$

This proves that

$$m(D'_q(i)) \ll R^{-\frac{\alpha\eta'q}{n^2}}.$$

This finishes the proof for the first case.

Now let us take care of the second case. Let us denote

$$\xi(t) := \begin{bmatrix} b^{-\beta t} & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & b^{r_{n_1}t} & \\ & & & & \ddots \\ & & & & & b^{r_n t} \end{bmatrix} \in \mathrm{SL}(n+1, \mathbb{R})$$

where  $\beta = \sum_{j=n_1}^n r_j < \eta$  and

$$g'(t) := \xi(t)g_{\mathbf{r}}(t) = \begin{bmatrix} b^{\chi t} & & & & \\ & b^{-r_1 t} & & & \\ & & \ddots & & \\ & & & b^{-r_{n_1-1}t} & \\ & & & & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix}$$

where  $\chi = \sum_{j=1}^{n_1-1} r_j$ . Then it is easy to see that

$$\xi(t)\mathbf{w}_+ = b^{-\beta t}\mathbf{w}_+,$$

$$\xi(t)\mathbf{w}_j = \mathbf{w}_j$$

for  $j = 1, \dots, n_1 - 1$ , and

$$\xi(t)\mathbf{w}_j = b^{r_j t}\mathbf{w}_j$$

for  $j = n_1, \dots, n$ . Then we have that

$$\begin{aligned} \|g'(q)U(\varphi(x))\mathbf{v}\| &= \|\xi(q)(\mathbf{w}_+ \wedge \mathbf{w}^{(i-1)} + \mathbf{w}^{(i)})\| \\ &\leq \|\xi(q)(\mathbf{w}_+ \wedge \mathbf{w}^{(i-1)})\| + \|\xi(q)\mathbf{w}^{(i)}\| \\ &\leq \|\mathbf{w}_+ \wedge \mathbf{w}^{(i-1)}\| + b^{\beta q}\|\mathbf{w}^{(i)}\| \\ &\leq \rho^i + b^{\beta q}R^{-2\eta'q}\rho^i \\ &\leq \rho^i + b^{\eta q}R^{-2\eta'q}\rho^i \leq \rho^i + R^{-\eta'q}\rho^i < (2\rho)^i. \end{aligned}$$

Moreover, for any  $x' \in \Delta(x) := [x - R^{-q(1-2\eta')}, x + R^{-q(1-2\eta')}]$ , we also have that

$$\|g'(q)U(\varphi(x'))\mathbf{v}\| < (2\rho)^i.$$

Let  $C > 0$  and  $\alpha > 0$  be the constants given in Theorem 5.1. We can choose  $0 < \rho < 1$  small enough at beginning such that there exists another small constant  $0 < \rho_1 < 1$  with  $C \left(\frac{2\rho}{\rho_1}\right)^\alpha < \frac{1}{1000}$ . Then by the Minkowski Theorem, the inequality above implies that for any  $x' \in \Delta(x)$ , the lattice  $g'(q)U(\varphi(x'))\mathbb{Z}^{n+1}$  contains a nonzero vector of length  $< 2\rho$ . Let  $\mathbf{v}_{x'} \in \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$  be the vector such that  $\|g'(q)U(\varphi(x'))\mathbf{v}_{x'}\| < 2\rho$ . Let us write

$$\mathbf{v}_{x'} = (v_{x'}(0), v_{x'}(1), \dots, v_{x'}(n)).$$

Then for  $j = n_1, \dots, n$ , we have that  $|v_{x'}(j)| < 2\rho$ . Therefore,  $v_{x'}(j) = 0$  for any  $j = n_1, \dots, n$ . In other words,  $\mathbf{v}_{x'}$  is contained in the subspace spanned  $\{\mathbf{w}_+, \mathbf{w}_1, \dots, \mathbf{w}_{n_1-1}\}$ . For notational simplicity, let us denote this subspace by  $\mathbb{R}^{n_1}$  and denote the set of integer points contained in the subspace by  $\mathbb{Z}^{n_1}$ . Accordingly, let us denote by  $\mathrm{SL}(n_1, \mathbb{R})$  the subgroup

$$\left\{ \begin{bmatrix} X & \\ & \mathrm{I}_{n+1-n_1} \end{bmatrix} : X \in \mathrm{SL}(n_1, \mathbb{R}) \right\} \subset \mathrm{SL}(n+1, \mathbb{R})$$

and denote by  $\mathrm{SL}(n_1, \mathbb{Z})$  the subgroup of integer points in  $\mathrm{SL}(n_1, \mathbb{R})$ . Note that  $g'(g) \in \mathrm{SL}(n_1, \mathbb{R})$  and  $U(\varphi(x'))$  can be considered as an element in  $\mathrm{SL}(n_1, \mathbb{R})$  since it preserves  $\mathbb{R}^{n_1}$ . Then  $\|g'(q)U(\varphi(x'))\mathbf{v}_{x'}\| < 2\rho$  implies that for any  $x' \in \Delta(x)$ , the lattice  $g'(q)U(\varphi(x'))\mathbb{Z}^{n_1}$  contains a nonzero vector of length  $< 2\rho$ . Let  $K_{2\rho}(n_1) \subset X(n_1) = \mathrm{SL}(n_1, \mathbb{R})/\mathrm{SL}(n_1, \mathbb{Z})$  denote the set of unimodular lattices in  $\mathbb{R}^{n_1}$  which do not contain any nonzero vector of length  $< 2\rho$ . Then the claim above implies that

$$m(\{x' \in \Delta(x) : g'(q)U(\varphi(x'))\mathbb{Z}^{n_1} \notin K_{2\rho}(n_1)\}) = m(\Delta(x)).$$

By Theorem 5.1, there exist  $j \in 1, \dots, n_1 - 1$  and  $\mathbf{v}' = \mathbf{v}'_1 \wedge \dots \wedge \mathbf{v}'_j \in \bigwedge^j \mathbb{Z}^{n_1} \setminus \{\mathbf{0}\}$  such that

$$\max\{\|g'(q)U(\varphi(x'))\mathbf{v}'\| : x' \in [x - R^{-q(1-2\eta')}, x + R^{-q(1-2\eta')}] \} < \rho_1^j$$

since otherwise we will have that

$$m(\{x' \in \Delta(x) : g'(q)U(\varphi(x'))\mathbb{Z}^{n_1} \notin K_{2\rho}(n_1)\}) \leq C \left(\frac{2\rho}{\rho_1}\right)^\alpha m(\Delta(x)) < \frac{1}{1000} m(\Delta(x)).$$

This reduces the second case in dimension  $n+1$  to the first case in dimension  $n_1 < n+1$ . Then we can finish the proof by induction.  $\square$

Now we are ready to prove Proposition 5.11.

*Proof of Proposition 5.11.* Recall that in Theorem 4.3, we denote by  $E_q$  the union of all  $q$ -extremely dangerous intervals. By Lemma 5.12, we have that

$$I_q \subset E_q \bigcup_{i=2}^n D_q(i).$$

By Theorem 4.3, we have that

$$m(E_q) \ll R^{-\nu q}$$

for some constant  $\nu > 0$ . On the other hand, by Lemma 5.14, we have that

$$m(D_q(i)) \ll R^{-\nu q}$$

for any  $i = 2, \dots, n$ . Therefore, we have that

$$\begin{aligned} F(\hat{\mathcal{I}}_{q,0}, I)R^{-q} &= m\left(\bigcup_{I_q \in \hat{\mathcal{I}}_{q,0}} I_q\right) \\ &\leq m\left(E_q \bigcup_{i=2}^n D_q(i)\right) \leq m(E_q) + \sum_{i=2}^n m(D_q(i)) \ll R^{-\nu q}. \end{aligned}$$

This completes the proof.  $\square$

Now we are ready to prove Proposition 3.7 for  $q > 10^6 n^4 N m$ .

*Proof of Proposition 3.7 for  $q > 10^6 n^4 N m$ .* We can choose  $M$  such that  $M^\nu > 1000$ . Recall that  $R \geq M$ . By Proposition 5.11, we have that

$$(5.4) \quad \left(\frac{4}{R}\right)^q F(\hat{\mathcal{I}}_{q,0}, I) \ll \left(\frac{4}{R}\right)^q R^{(1-\nu)q} = \left(\frac{4}{R^\nu}\right)^q < \left(\frac{4}{1000}\right)^q.$$

Combining (5.1), (5.2) and (5.4), we have that

$$\sum_{p=0}^{q-1} \left(\frac{4}{R}\right)^{q-p} \max_{I_p \in \mathcal{I}_p} F(\hat{\mathcal{I}}_{q,p}, I_p) \rightarrow 0$$

as  $m \rightarrow \infty$ . This proves the statement.  $\square$

## REFERENCES

- [An16] Jinpeng An. 2-dimensional badly approximable vectors and schmidt's game. *Duke Math. J.*, 165(2):267–284, 02 2016.
- [Ber15] Victor Beresnevich. Badly approximable points on manifolds. *Inventiones mathematicae*, 202(3):1199–1240, 2015.
- [BKM01] V Bernik, D Kleinbock, and Grigorij A Margulis. Khintchine-type theorems on manifolds: the convergence case for standard and multiplicative versions. *International Mathematics Research Notices*, 2001(9):453–486, 2001.
- [BPV11] Dzmitry Badziahin, Andrew Pollington, and Sanju Velani. On a problem in simultaneous diophantine approximation: Schmidt's conjecture. *Annals of mathematics*, 174(3):1837–1883, 2011.
- [Cas57] J. S. Cassels. *An introduction to Diophantine approximation*. Cambridge University Press, 1957.
- [Dan84] SG Dani. On orbits of unipotent flows on homogeneous spaces. *Ergodic Theory and Dynamical Systems*, 4(01):25–34, 1984.
- [Dan86] SG Dani. Bounded orbits of flows on homogeneous spaces. *Commentarii Mathematici Helvetici*, 61(1):636–660, 1986.
- [Jar28] Vojtěch Jarník. Zur metrischen theorie der diophantischen approximationen. 1928.
- [KM96] DY Kleinbock and GA Margulis. Bounded orbits of nonquasiunipotent flows on homogeneous spaces. *American Mathematical Society Translations*, pages 141–172, 1996.
- [KM98] Dmitry Y Kleinbock and Grigorij A Margulis. Flows on homogeneous spaces and diophantine approximation on manifolds. *Annals of Mathematics*, pages 339–360, 1998.
- [KW08] Dmitry Kleinbock and Barak Weiss. Dirichlet's theorem on diophantine approximation and homogeneous flows. *Journal of Modern Dynamics*, 2(1):43–62, 2008.
- [KW10] Dmitry Kleinbock and Barak Weiss. Modified schmidt games and diophantine approximation with weights. *Advances in Mathematics*, 223(4):1276–1298, 2010.
- [LM14] Elon Lindenstrauss and Gregory Margulis. Effective estimates on indefinite ternary forms. *Israel Journal of Mathematics*, 203(1):445–499, 2014.
- [Mah39] Kurt Mahler. Ein übertragungsprinzip für lineare ungleichungen. *Časopis pro pěstování matematiky a fysiky*, 68(3):85–92, 1939.
- [MS95] Shahar Mozes and Nimish Shah. On the space of ergodic invariant measures of unipotent flows. *Ergodic Theory and Dynamical Systems*, 15(01):149–159, 1995.
- [MT94] GA Margulis and GM Tomanov. Invariant measures for actions of unipotent groups over local fields on homogeneous spaces. *Inventiones mathematicae*, 116(1):347–392, 1994.
- [PV02] Andrew Pollington and Sanju Velani. On simultaneously badly approximable numbers. *Journal of the London Mathematical Society*, 66(1):29–40, 2002.
- [Rat91] Marina Ratner. On raghunathan's measure conjecture. *Annals of Mathematics*, pages 545–607, 1991.

- [Sch66] Wolfgang M Schmidt. On badly approximable numbers and certain games. *Transactions of the American Mathematical Society*, 123(1):178–199, 1966.
- [Sch83] Wolfgang M Schmidt. Open problems in diophantine approximation. *Diophantine approximations and transcendental numbers (Luminy, 1982)*, 31:271–287, 1983.
- [Sha09a] Nimish A Shah. Equidistribution of expanding translates of curves and dirichlet’s theorem on diophantine approximation. *Inventiones mathematicae*, 177(3):509–532, 2009.
- [Sha09b] Nimish A. Shah. Limiting distributions of curves under geodesic flow on hyperbolic manifolds. *Duke Mathematical Journal*, 148(2):251–279, 2009.
- [Sha10] Nimish Shah. Expanding translates of curves and dirichlet-minkowski theorem on linear forms. *Journal of the American Mathematical Society*, 23(2):563–589, 2010.

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